

CALCULUS AND DIFFERENTIAL EQUATIONS

MATH 1B

Lecture 27: Stability of autonomous systems

AUTONOMOUS SYSTEMS

27.1. A differential equation of the form

$$y' = f(y)$$

is called an **autonomous differential equation**. An important example is the **Malthusian system** $y' = ry$. It produces exponential growth for $r > 0$ and exponential decay for $r < 0$. An other example is the **logistic system** $y' = ry(1 - \frac{y}{K})$. A third example is the general linear system $y' = ry + a$ which appears in **banking problems** or in **input-output models** or **compartmental problems**.

EQUILIBRIA

27.2. We are interested in solutions $y(t)$ which do not depend on time. For autonomous systems, this happens if y is a root of f . This is called an **equilibrium**. Equilibria are important because if y is an equilibrium, then the quantity y is stationary there. Looking at equilibrium points vastly helps to understand an autonomous system.

27.3.

Example: The **exponential model** $y' = ry$ has only one equilibrium. It is $y = 0$.

Example: The **logistic model** $y' = ry(1 - \frac{y}{K})$ has two equilibria, $y = 0$ and $y = K$.

Example: The **input-output model** $y' = ry + a$ has the equilibrium $y = -a/r$.

27.4.

Equilibria solutions are solutions to the differential equation that are constant in time.

It also helps to remember the solution of an input-output model. The general solution is $y(t) = -a/r + Ce^{rt}$ is the sum of the equilibrium solution $y = -a/r$ and the general solution Ce^{rt} of the exponential model.

STABILITY

27.5. It is useful to know whether an equilibrium is **stable** or **unstable**. An equilibrium y_0 is called stable, if $y(t) \rightarrow y_0$ in a neighborhood of y_0 . If a solution is not stable it is called **unstable**. The equilibrium $y = 0$ for example is stable for the system $y' = ry$ if r is negative and unstable if r is positive. Because near an equilibrium point c the function has a Taylor expansion $f(y) = r(y-c) + b(y-c)^2/2 + \dots$ and higher order terms can be neglected, the stability is entirely determined by the value $r = f'(c)$.

If c is an equilibrium and $f'(c) < 0$, then c is stable. If $f'(c) > 0$, it is unstable. If $f'(c) = 0$, the stability needs to be decided otherwise.

GRAPHICAL ANALYSIS

27.6. In order to see this graphically, draw the graph $f(y)$. Note that this is a picture in the $y - y'$ **plane**. It has nothing to do with the $t - y$ **plane** we have looked at when we were discussing **slope fields**. Do not mix up these two pictures. There is no time axis in the $y - y'$ picture.

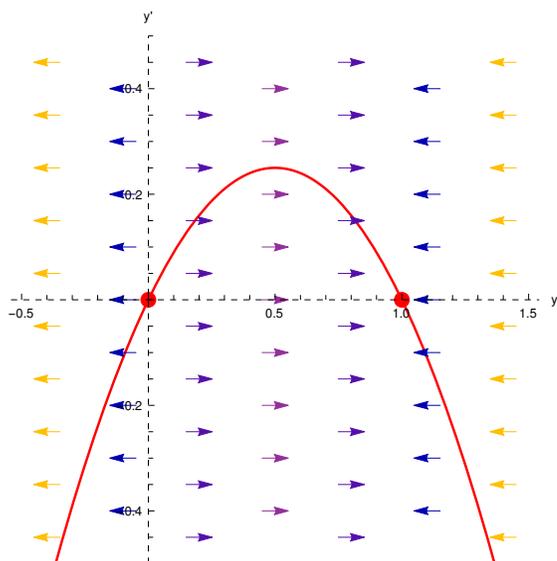


FIGURE 1. The $y - y'$ **plane** in which one can plot the graph of the function $f(y) = y(1 - y)$ and the equilibrium points. If $f(y)$ is negative (like $y = 1$ here), we move towards the point near the equilibrium. The equilibrium is stable. On the other hand, the equilibrium point 0 is unstable. The solution moves away from it.