DIFFERENTIAL GEOMETRY

MATH 136

Final Exam Paper

Assignment: due December 15

Write an expository paper about a topic in differential geometry of your choice.

Rules

- The paper must be written by yourself. No human or AI assistance is allowed in writing.
- Acknowledge any references used: books, papers, web, AI chat, discussions, the use of computer algebra.
- Aim for 4 or more pages with references. There can be illustrations. Also for illustrations, either do them yourself or give credit.

Grading criteria

(1) Mathematical correctness
(2) Clarity, readability and elegance
(3) References and sources
(4) Adaptation to the course, notation
(5) Originality or depth or surprise

We will give 10 points for 1) and 2), 10 points for 3) and 4) and 5 points for 5) The quiz gives 5 points. There are aximally 30 points.

Topic suggestions

Pick or modify one of the following topics. If you prefer an other topic you need to check with Oliver.

(1) **"My favorite manifold"**. i.e. projective plane, the Klein bottle, the projective 3-space $SO(3)$ (a 3 manifold) or $SU(3)$ (a 8 dimensional manifold).
(2) **"The world of discrete manifolds"**. Explore small examples of discrete manifolds, I.e. Klein bottles, projective planes, or higher genus Klein bottles.
(3) **"The Hilbert Action"** Outline the proof that critical points of the Hilbert action are Einstein manifolds.
(4) **"What are Minimal Surfaces?"**. See chapter 3D in Kuehnel.
(5) **"The rigidity of the sphere"** a compact connected regular surface of constant curvature must be a sphere. There is a section in Do Carmo.
(6) **"What are Ruled surfaces?"** Section 3C in Kuehnel.

(7) **"The amazing Moebius strip"**. Explore differential geometry of Moebius band embedded in $\mathbb{R}^3$.
(8) **"Clairaut's relation for geodesics"** A conserved quantity for geodesics in surfaces of revolution".
(9) **"The isoperimetric inequality"**. There is an exposition in Do Carmo.
(10) **"Geodesic rigidity of Gauss"**: if p, q are two points on a manifold and if $\gamma_s(t)$ is a smooth family of geodesics connecting p, q, then all these geodesics must have the same length.
(11) **"The Willmore energy"** The boy surface is known as "Oberwohlfach surface". A theorem of Bryant-Kusner tells that the Boy surface minimizes the Willmore energy.
(12) **"Geometry in hyperbolic space"**. Describe the geodesics in a 2 dimensional manifold of constant curvature -1. Gauss-Bonnet for polygons.
(13) **"Positive Curvature Manifolds"**. The zoo of even dimensional manifolds for which all sectional curvatures are positive.
(14) **"Fenchel's theorem"** or the Fary-Milnor theorem about total curvature. There is a section in Kuehnel.
(15) **"The fundamental theorem of Riemannian geometry"** a connection satisfying tree axioms must be the Christoffel connection.
(16) **"Classifying 2-manifolds"**. Describe the classification of 2 manifold as a connected sum of tori or projective planes.
(17) **"Spherical space forms"**. See for example Theorem 7.30 in Kuehnel.
(18) **"What are lense spaces"**? What are spherical 3-manifolds?
(19) **"Synge's theorem"**: an eve dimensional orientable positive curvature manifold is simply connected.
(20) **"The Poinare conjecture"**. How was the Poincare conjecture proven. Especially define the Ricci flow using notation we have used.
(21) **"Surgery of 3-manifolds"**. Explain how one can use knots to build 3-manifolds. Especially Dehn surgery and the Lickorish-Wallace theorem.
(22) **"Exotic spheres?"** What are spheres with non-standard differential structure.
(23) **"The Kerr Metric"**. A solution of the Einstein equations. The Kerr metric" models a rotating non-charged black hole. The Kerr-Newman metric a rotating charged black hole.
(24) **"Bertrand-Puiseux formulas"** seen in Unit 1. Prove $|S_r(p)| = 2\pi r - \pi K \epsilon^3/3 + ...$
(25) **"Chaotic Billiards"**. The geodesic flow in manifolds with boundary. is called a billiard. This involves curvature.
(26) **"Riemannian metrics on Lie groups"**. Like on $SO(3)$ or $SU(2)$. How does one get a Riemannian metric on such spaces?
(27) **"The Willmore conjecture"**. What is the Willmore energy? Explain the conjecture and how it was solved.
(28) **"The Differentiable Jordan Curve theorem"** is a bit easier than the actual Jordan Curve theorem in topology. Prove it. See Do carmo page 400.
(29) **"The Hopf-Rynov Theorem"**: Two points on a complete surface can be joined by a minimal geodesic. See Do Carmo page 338.
(30) **"The formulas of Codazzi-Meinardi"**. See section 4.C in Kuehnel.

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