DIFFERENTIAL GEOMETRY

MATH 136

Unit 0: Introduction

0.1. This is an introduction to the Riemannian geometry of curves, surfaces and manifolds. We will also develop some less technical discrete differential geometry and see how the mathematics is applied to general relativity. Most differential geometry texts go too far for a 20 lecture course in which the goal is to prove Gauss-Bonnet in full detail using a multi-variable calculus and linear algebra background only.

0.2. Many texts in Riemannian geometry already early try to go "coordinate free". This is more elegant but also more abstract. It can come with frustrations. How can one "see" or "work" with an axiomatically defined connection for example? We opted for a concrete approach in which everything can be computed explicitly, either by hand or by using a few lines of a symbolic algebra. A few lines are needed only to compute the Einstein tensor of a manifold. Geodesics are directly introduced using the Euler-Lagrange variational principle. Code provided to visualize these geodesics in an arbitrary given manifold.

0.3. Low dimensional Riemannian geometry is an extremely active area of mathematics. There are many open problems and applications. There are close relations to computer science and computer graphics. Taking the computer science point of view and sticking to coordinates is something, which many mathematicians scoff about. It is true that a coordinate free approach can feel more elegant, but it can also be just dream walking and talk about objects one has no intuition about. Especially for a beginner, it is good to be able to be able to see everything in a concrete manner at first. Going abstract afterwards is much easier.

0.4. Riemannian geometry is not only a prototype mathematical theory, it also helps to inspire other fields like topology or combinatorics. Topological statements like that a simply connected 3-manifold is topologically equivalent to a sphere was solved by adding a Riemannian metric structure on the manifold first. Combinatorial notions of manifolds have led to interesting combinatorial questions. We will see here how easy the discrete set-up is. A precise frame work for discrete manifolds including all differential geometry can be done in a few lines. Gauss-Bonnet for general networks is almost a joke when compared to the difficulties we encounter in the continuum.

0.5. From the application side, the subject is a marvel. The geometry of Riemannian manifolds is the lingua franca of gravity. Riemannian manifolds are also an inspiration for the arts. The actual metric implementation of a manifold is important in aesthetics. Both very smooth or polyhedral implementations can matter. When looking at the shape of objects like cars, houses, cloths or tools, one can see oscillations over time between smooth and edgy. Geometric considerations also matter more and more in data analysis. Artificial intelligence frame work embed knowledge in higher dimensional spaces and use distances to build large language models.

0.6. The subject is also saturated with unsolved problems. Simple sounding questions are unresolved like whether a positive curvature even dimensional manifold has positive Euler characteristic or whether there is a positive curvature metric on the product of two 2-spheres. We have even in the case of ellipsoids no answer yet about the number of cusps of caustics of wave fronts. How many closed geodesics are there for a given manifold? What are the Wiederkehr manifolds in higher dimensions, manifolds where all geodesics are closed. For which type of manifolds do spectral properties determine the manifold? Are two manifolds with isomorphic geodesic flows and equal diameter automatically isometric?

0.7. We had 20 lectures for this course. The goal was to reach four mountain peaks:

(1) the Frenet-Serret theorem
(2) the Gauss-Bonnet theorem
(3) the Theorema egregium
(4) the Einstein equations

We restrict to 2 pages per lecture because two pages are realistically teachable in an hour. The first three every first course in Riemannian geometry covers, the last has been reached sometimes here in the past. Unlike the first course in Riemannian geometry I was exposed when I was in college, we do not restrict to dimension 2. But Gauss-Bonnet is for 2-manifolds. The higher dimensional Gauss-Bonnet-Chern theorem is almost never proven, even in graduate courses.

0.8. Some literature consulted:

- W. Kühnel, Differential Geometry: Curves - Surfaces - Manifolds", 3. Edition.
- M. P. Do Carmo, Differential Geometry of Curves and Surfaces", 2. Edition. [1]
- M. Berger, "A panoramic view of Riemannian geometry". A marvel.
- S. Gudmundsson, "An introduction to Gaussian Geometry". (Nice 2023 notes).
- I.A. Taimanov, "Lectures on Differential geometry". (Refreshingly short).
- H.L. Cycon, R.G. Froese, W.Kirsch, B.Simon: "Schrödinger operators". (Ch.12).
- J.A. Thorpe, "Elementary Topics in Differential geometry".
- A. Pressley, "Elementary Differential geometry". (2. edition).
- M. Lipschultz, "Schaum Outline: differential Geometry".
- C.W. Misner, K.S. Thorne, and J.A. Wheeler "Gravitation".
- Y.Choquet-Bruhat: "General relativity and the Einstein equations".
- J.A. Wheeler: "Journey into gravity and space-time".
- Y. Choquet Bruhat and C. DeWitt-Morette: "Analysis, Manifolds and Physics".
- J. Marsden and T. Ratiu: "Manifolds, Tensor Analysis".
- O. Knill, "Introduction to geometry and geometric analysis". (1995 notes).

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[1] Both texts have been used traditionally in the last 20 years here at Harvard

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Unit 1: What is differential geometry?

Introduction

1.1. Differential geometry deals with geometric objects called **manifolds**. Manifolds can be described intrinsically. It is convenient however to look first at manifolds embedded in a Euclidean space $\mathbb{R}^n$ like our space $\mathbb{R}^3$. Examples are one dimensional manifolds called curves or two dimensional manifolds called surfaces or hypersurfaces given as a root of a function f. The 3-sphere $x_1^2 + x_1^2 + x_3^2 + x_4^2 = 1$ for example is a 3-dimensional manifold embedded in 4-dimensional space. Since we live in a 3-dimensional space, we are at first mostly interested in one or two dimensional manifolds in $\mathbb{R}^3$. These objects are called **curves** or **surfaces**.

1.2. A **curve** is given as a **parametrization** $r(t) = [x(t), y(t), z(t)]$ where $t \in [a, b]$ is the parameter domain and $x(t), y(t), z(t)$ are functions of one variable. In two dimensions, some curves can also be written implicitly as a **level curve** like for example the circle $x^2 + y^2 = 1$. A **surface** is given as a parametrization $r(u, v) = [x(u, v), y(u, v), z(u, v)]$ where (u, v) is in some domain $R \subset \mathbb{R}^2$ in the uv-plane. Some surfaces can be given as **level surfaces** $f(x, y, z) = 0$. Some curves can be given as intersection of level surfaces $f(x, y, z) = 0, g(x, y, z) = 0$.

1.3. We are interested in **global quantities** like **arc length** $\int_a^b |r'(t)|\, dt$ or **surface area** $\iint_R |r_u \times r_v|\, dudv$, where $\times$ is the **cross product** as well as **local quantities** like **curvature** $\kappa(t) = |r'(t) \times r''(t)|/|r'(t)|^3$ or **torsion** $\tau(t) = \det[r'(t), r''(t), r'''(t)]/|r' \times r''|^2$ for curves. For a surface, the **curvature** of a point can be defined as $K(p) = \lim_{r\to 0} 3\frac{2\pi r - |S_r(p)|)}{\pi r^3}$, where $|S_r(p)|$ is the length of the **wave front** $S_r(p)$ of points on the surface in distance r from p which is a circle for small r. This is an intrinsic definition not making use of the embedding of the surface in an ambient space. It even makes sense on non-smooth surfaces, like polyhedra.

1.4. Curvature plays an important role in differential geometry. We will define it differently later in the course and verify that it is independent of the embedding in space. This is the **Theorema egregium**, the "great theorem" of Gauss from 1827. Riemannian geometry, the idea of doing geometry on a manifold without having to embed it into an ambient emerged in an inaugural lecture of Riemann in 1854. The theory is used heavily in Einstein's 1915 theory of general relativity to which Schwarzschild found a black hole solution in 1916. A 100 years later, gravitational waves produced by black hole mergers have been observed.

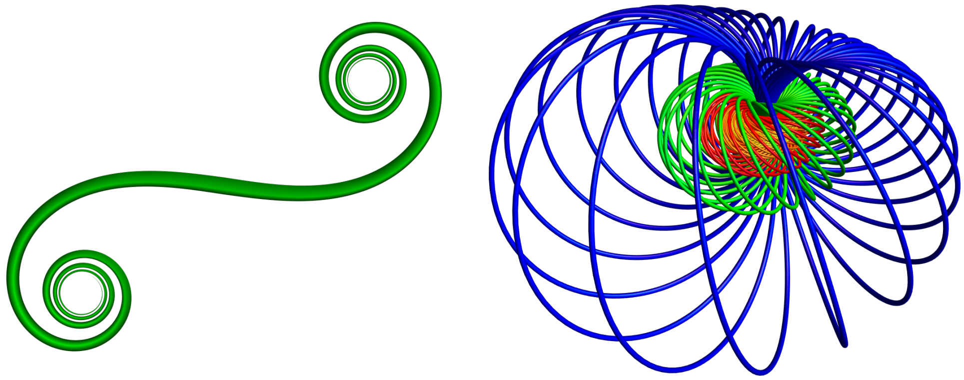

FIGURE 1. To the left we see a curve called Euler curve, to the right, we see a visualization of the 3 dimensional sphere $x^2 + y^2 + z^2 + w^2 = 1$ in $\mathbb{R}^4$. It is foliated by 2 dimensional flat tori. We can not embed flat tori in $\mathbb{R}^3$ but we can in $\mathbb{R}^4$.

1.5. The focus in differential geometry is to investigate relations between local quantities and global properties like for example to integrate up curvature. We are especially interested in quantities that do not depend on the metric, like the **Euler characteristic**. Here are examples which will appear early in this course: for a planar closed curve in $\mathbb{R}^2$ one can define the **signed curvature** as $\kappa(t) = (r'(t) \times r''(t))/|r'(t)|^3$. We will then see the **Hopf Umlaufsatz** $\int_C \kappa(t)\, dr(t) = 2\pi$. For a two-dimensional surface with g holes of Euler characteristic $\chi(G) = 2 - 2g$, one has the **Gauss-Bonnet theorem** $\iint_R K(x)\, dV(x) = 2\pi\chi(G)$. We also want to understand **geodesics**, curves that locally minimize length. One can start geodesics into any direction $v/|v|$ and let it run for a distance $|v|$. This produces the **exponential map** $\exp_p$, a map from the tangent space T_pM of a point p to the manifold. If S_r is the sphere of radius r in $\mathbb{R}^2$, then the image $W_r(t) = \exp_p(S_r)$ is called the **wave front** at p. These waves can become complicated for large r. This can also be studied on polyhedra. We expect wave fronts to become dense in the manifold, except for very special cases like the round sphere.

1.6. Differential geometry then extends curve and surface theory to arbitrary dimensions. One study then so called **Riemannian manifolds** or **pseudo-Riemannian manifolds** which appear in physics. There is an intrinsic geometry but also interest when manifolds M are embedded in larger manifolds M'. In general relativity for example, space is a 3-dimensional manifold embedded in a four dimensional **space-time manifold** M'. The above formulation of curves or surfaces dealt with embeddings of one or two dimensional manifolds in Euclidean 3-manifold M'. One can use the exponential map to define **sectional curvature** and to use it to define a **curvature tensor** or **scalar curvature**. The extrema of the functional that gives the total scalar curvature are the Einstein equations. **General relativity** studies solutions of these equations as they tell how matter bends space. The geodesic equations then tell, how matter moves in this space.

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Unit 2: Some analysis

1.1. A map

$$f : \mathbb{R}^m \to \mathbb{R}^n, \begin{bmatrix} x_1 \\ \dots \\ x_m \end{bmatrix} \mapsto \begin{bmatrix} f_1(x_1, \dots x_m) \\ \dots \\ f_n(x_1, \dots, x_m) \end{bmatrix}$$

is called **differentiable** or C^1, if all derivatives $\frac{\partial}{\partial x^j} f_i$ are continuous. For such a map, define the $n \times m$ **Jacobean matrix**

$$df(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} f_1(x_1, \dots, x_m) & \cdots & \frac{\partial}{\partial x_m} f_1(x_1, \dots, x_m) \\ \dots & & \\ \frac{\partial}{\partial x_1} f_n(x_1, \dots, x_m) & \cdots & \frac{\partial}{\partial x_m} f_n(x_1, \dots, x_m) \end{bmatrix} .$$

1.2. a) If $r(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ is a curve, then $dr(t) = r'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$ is the **velocity**.
b) If $f(x, y, z)$ is a function of 3 variables, then $df(x, y, z) = [f_x, f_y, f_z]$ is a 1×2 matrix. It is the transpose of the gradient.
c) For a vector field $f(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}$, we have $df = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$. It is used also when analyzing systems of differential equations $x' = f(x)$.
d) If $f(x) = Ax$ is a linear map, then $df(x) = A$.

1.3. We say df has **maximal rank** if its rank is the minimum of m, n. If $m < n$, then the image of f is in general a m-dimensional set in $\mathbb{R}^n$. If $n < m$, then we can look at the roots $f(x) = 0$ which is a $n - m$ dimensional set. If $m \leq n$, we can say that f is **manifold like** near $f(x)$ if $df(x)$ has maximal rank. If $n \leq m$, then $M = \{f - c = 0\}$ is **manifold like** near x, if $df(x)$ has maximal rank.

1.4. Lets look at the case $m = 2, n = 1$, where $f(x, y)$ is C^1 meaning both partial derivatives $f_x(x, y)$ and $f_y(x, y)$ are continuous. Let us look at a curve

$$f(x, y) = x^3y^2 - xy^3 = -4 .$$

We are interested in how this curve looks like near the point $(1, 2)$. To investigate this, we look at the Jacobean matrix $df(x, y) = [3x^2y^2 - y^3, 2x^3y - 3xy^2]$. At the point $(1, 2)$, this is $[4, -8]$. The line $4x - 8y = -12$ is tangent to the curve. If we write the curve as $y = g(x)$ near $x = 1$, then $f(x, g(x)) = -4$ and differentiating gives $f_x + f_y g' = 0$ so that $g' = -f_x/f_y$.

Theorem 1 (Implicit function theorem)**.** *If $f(x,y) = f(x_0, y_0) = c$ and $f_y(x_0, y_0) \neq 0$, then $f(x,y) = c$ can near (x_0, y_0) be written as $y = g(x)$ for some C^1 function $g(x)$.*

Proof. Take a small neighborhood $U = I \times J$ of (x_0, y_0) where $|f_y(x,y)| \geq c$ and $|f_x(x,y)| \leq d$. Given $(x,y) \in U$ and $f(x_0, y_0) = 0$ and $y \to f_y(x,y)$ is bounded away from 0, we have $f(x, y_0 - t) f(x, y_0 + t) < 0$ and by the **intermediate value theorem**, there exists t such that $f(x, y_0 + t) = 0$. This gives us a function $g(x) = y_0 + t$ and $f(x,y) = c$ agrees with $y = g(x)$ in U. By the chain rule $f_x 1 + f_y g' = 0$ we see that g is differentiable with $g' = -f_x/f_y$ and $|g'(x)| \leq d/c$ in U. ☐

1.5. It follows that if a C^1 function f is manifold like near (x_0, y_0) then the level set $f(x,y) = c$ is near (x_0, y_0) the graph of a function. If $f_y \neq 0$ use the implicit function theorem as stated and $y = g(x)$. If $f_y = 0$, we must have $f_x \neq 0$ and $f(x,y) = c$ is the graph of $x = g(y)$ for some C^1 function g.

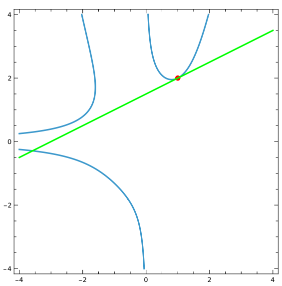

FIGURE 1. $f(x,y) = c$ can near (x_0, y_0) be written as a graph $y = g(x)$.

1.6. If $m < n$, then f has maximal rank if df has rank m. We look at this case more next week and look at $g = df^T df$. Lets look at a C^1 curve C defined by the parametrization $r(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. Its velocity if $dr = r'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix}$. If the velocity is not zero at $t = t_0$ then the curve is manifold like. The curve is then close to the line $l(s) = r(t_0) + s r'(t_0)$. If $r'(t_0) \neq 0$, then C is a graph $(s, g(s))$ close to the line $(s, l(s)$.

1.7. In the case $m = n$, the map $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field or **coordinate change**. The image of f is manifold like near x if df has maximal rank meaning that the kernel is zero meaning that the determinant of df is non-zero. In that case f is invertible near x. This is called the **inverse function theorem**.

Theorem 2 (Inverse function theorem)**.** *If $h(y) \in C^1$ has a non-zero derivative at y_0, then h is invertible near y_0 and $y = g(x)$ near x_0. g is C^1 with $dg(x_0) = dh(y_0)^{-1}$.*

Proof. Define $f(x,y) = x - h(y) = 0$. As $f_y = h'(y)$ is non-zero, the above theorem applies and there exists a function $g(x)$ such that $y = g(x)$ near x_0. ☐

All proofs can be generalized to maps $f : \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^n \to \mathbb{R}^n$, $(x,y) \mapsto f(x,y) \in \mathbb{R}^n$. Now f_x is a $n \times k$ matrix and f_y is a $n \times n$ matrix. If f_y is invertible, then df has maximal rank and $f(x,y) = c$ can be written as $y = g(x)$ and $n \times k$ matrix $dg = -f_y^{-1} f_x$. This is the implicit function theorem. If $k = 0$, where $f : \mathbb{R}^n \to \mathbb{R}^n$, then $df(x)$ invertible implies f is invertible near x. This is the inverse function theorem.

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Unit 2: Surfaces

2.1. Geometric objects can be given as **level sets**, kernels $\{f = 0\}$ of smooth maps $f : \mathbb{R}^m \to \mathbb{R}^n$ with $n < m$ or **parametrizations**, images of smooth maps f from a subset R of $\mathbb{R}^m \to \mathbb{R}^n$ with $m < n$. If in the level set case, df has maximal rank n everywhere, we get a **manifold**. [1] The same happens in the parametrization case, if f is injective and df has maximal rank m everywhere.

2.2. An example of **level surface** $\{f = 0\}$ of a function $f : \mathbb{R}^3 \to \mathbb{R}$ with $df \neq 0$ everywhere is the sphere $x^2 + y^2 + z^2 - 1 = 0$. An other example is a curve, the image of an interval $[a, b]$ to $\mathbb{R}^n$. The duality between kernel and image manifests already in linear algebra. The **kernel** $\ker(A)$ of a matrix A is the linear space $\{Ax = 0\}$ The **image** $\mathrm{im}(A)$ is the linear space $\{Ax\}$. The **fundamental theorem of linear algebra** is the wonderful duality $\boxed{\mathrm{im}(A^T) = \ker(A)^\perp}$.

Theorem: The image of A^T is perpendicular to the kernel of A.

Contour surfaces

2.3. If $f : \mathbb{R}^n \to \mathbb{R}$ is given, then the solutions of $f(x_1, \cdots, x_n) = d$ is called **hyper surface** or simply **surface** if $n = 3$. If the Jacobian matrix df (or equivalently the gradient $\nabla f = df^T$ is non-zero (meaning has maximal rank at every point), then $f = d$ is an example of a manifold. We will give more definitions later.

2.4. The case $f(x) = Ax$ is a hyperplane. **Quadratic manifolds** are $f(x) = x \cdot Bx + Ax = d$, where B is a symmetric matrix, A is a row vector and $d \in \mathbb{R}$ and df has maximal rank. Write $\mathrm{Diag}(a_1, \ldots, a_n)$ for diagonal and 1 for the identity matrix.

2.5. Examples: For $B = 1$ and $A = 0$ and $d = 1$ we get the **sphere** $|x|^2 = 1$. For $B = \mathrm{Diag}(1/a^2, 1/b^2, 1/c^2)$ is $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ we get ellipsoids. For $B = \mathrm{Diag}(1, 1, -1)$ and $d = 1$, we get a **one-sheeted hyperboloid** $x^2 + y^2 - z^2 = 1$. For $B = \mathrm{Diag}(1, 1, -1)$ and $d = -1$, we get a **two-sheeted hyperboloid** $x^2+y^2-z^2 = -1$. For $B = \mathrm{Diag}(1, 1, 0)$ and $A = [0, 0, -1]$ and $d = 0$ we get the **paraboloid** $x^2+y^2 = z$, for $B = \mathbf{Diag}(1, -1, 0)$ and $A = [0, 0, -1]$ and $d = 0$ we get the **hyperbolic paraboloid** $x^2 - y^2 = z$. We can recognize paraboloids by intersecting with $x = 0$ or $y = 0$ to see parabola. If $B = \mathrm{Diag}(1, 1, -1)$ and $d = 0$, we get a **cone** $x^2 + y^2 - z^2 = 0$. For $B = \mathrm{Diag}(1, 1, 0)$ and $d = 1$ we get the **cylinder** $x^2 + y^2 = 1$.

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[1] A theorem of Nash assures that every m-manifold can be embedded in some $\mathbb{R}^n$.

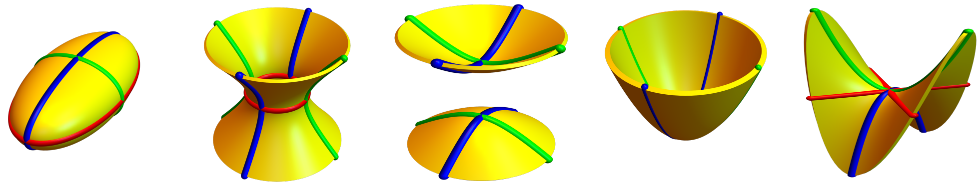

Figure 1. Ellipsoid, hyperboloids and paraboloids.

Parametrizations

2.6. A map $r : \mathbb{R}^m \to \mathbb{R}^n$ is called a **parametrization**. It is custom to use the letter r here, rather than f. We take the case $m < n$ and especially $m = 2, n = 3$. A map r from $\mathbb{R}$ to $\mathbb{R}^n$ is a **curve**. The image of a map $r : R \subset \mathbb{R}^m \to \mathbb{R}^n$ is then a **m-dimensional surface** in $\mathbb{R}^n$.

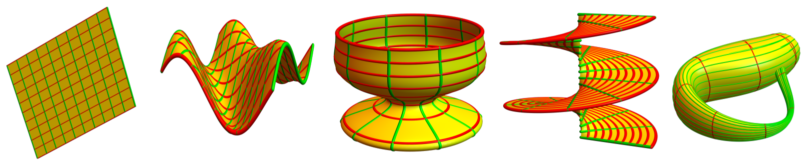

Figure 2. A plane, graph, surface of revolution, helicoid and Klein bottle

2.7. The parametrization $r(\phi, \theta) = [\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)]$ produces the **sphere** $x^2 + y^2 + z^2 = 1$. The full sphere uses $0 \leq \phi \leq \pi$, $0 \leq \theta < 2\pi$. By modifying the coordinates, we get an **ellipsoid** $r(\phi, \theta) = [a\sin(\phi)\cos(\theta), b\sin(\phi)\sin(\theta), c\cos(\phi)]$ satisfying $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. By allowing a, b, c to be functions of ϕ, θ we get "bumpy spheres" like $r(\phi, \theta) = (3 + \cos(3\phi)\sin(4\theta))[\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)]$.

2.8. If $r : \mathbb{R}^m \to \mathbb{R}^n$ with $m < n$ is a parametrization, then **Jacobian matrix** $dr(x)$ produces the $m \times m$ matrix with $\boxed{g = dr^T dr}$. It is the **first fundamental form**. For a parametrization $R : \mathbb{R}^2 \to \mathbb{R}^3$, the 3×2 matrix $dr(u, v)$ contains the vectors $\partial_u r = r_u, \partial_v r = r_v$ as columns and g is a 2×2 matrix.

2.9. The number $\boxed{|dr| = \sqrt{\det(dr^T dr)}}$ is called the **volume distortion factor**. The integral $\boxed{\int_R |dr(x)|\, dx}$ is the m-dimensional volume of the images $r(R) \subset \mathbb{R}^n$.

2.10. For a surface in $\mathbb{R}^3$, the surface area is $\boxed{\iint_R |r_u \times r_v|\, dudv}$ because

Theorem: $\det(dr^T dr) = |r_u \times r_v|^2$ for $r : \mathbb{R}^2 \to \mathbb{R}^3$.

Proof. As $dr^T dr = \begin{bmatrix} r_u \cdot r_u & r_u \cdot r_v \\ r_v \cdot r_u & r_v \cdot r_v \end{bmatrix}$, the identity is the **Cauchy-Binet identity** $|r_u \times r_v|^2 = |r_u|^2 |r_v|^2 - |r_u \cdot r_v|^2$ which boils down to $\sin^2(\theta) = 1 - \cos^2(\theta)$, where θ is the angle between the tangent vectors r_u and r_v. $\square$

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Unit 3: Curves

3.1. **Curves** in $\mathbb{R}^n$ can be be either given as images of smooth maps $r : \mathbb{R} \to \mathbb{R}^n$ or as solutions $f = 0$ to $(n-1)$ equations $f : \mathbb{R}^n \to \mathbb{R}^{n-1}$. In the first homework, you have seen the intersection of $n-1 = 2$ surfaces $f_1 = 0$ and $f_2 = 0$ in $\mathbb{R}^3$ which gave the Viviani curve. Looking at **solution sets of equations** is more like a **algebraic geometry** thing. Here, in differential geometry, we primarily look at **parametrizations** $[a, b] \to \mathbb{R}^n$. An example of a curve is the **helix** $r(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$ in $\mathbb{R}^3$. [1]

3.2. The **Jacobean matrix** of a curve $r(t)$ is $\boxed{dr(t)}$

$$r(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix}, dr(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \cdots \\ x_n'(t) \end{bmatrix} = r'(t) \ .$$

It is also known as the **velocity** and just abbreviated $\boxed{r'(t)}$. The **first fundamental form** is

$$g = dr^T dr = |r'(t)|^2 \ .$$

It is the square of the speed. The **arc length** of the curve is defined as

$$L = \int_a^b |r'(t)| \ dt \ .$$

Related to arc length is the **action**

$$I = \int_a^b |r'(t)|^2 \ dt \ .$$

which has the advantage that it can be computed better and produces equivalent variational problems. Minimizing the arc-length is equivalent to minimize the action and leads to geodesics. Here we are in flat Euclidean space and geodesics are straight lines. We will say more about this in class. You show in the homework:

Theorem 1 (Archimedes)**.** *The straight line is the shortest path connecting $A, B \in \mathbb{R}^n$.*

[1] We will often write also just $[\cos(t), \sin(t), t]^T$ or simply $[\cos(t), \sin(t), t]$ without the transpose for typographic reasons.

3.3. A curve is called **simple** if r is it does not have self intersections. It is called **regular** if the first fundamental form is nowhere zero. Equivalently, this means that the velocity is nowhere zero. A simple closed curve in space is called a **knot**. An example is the **figure 8 knot**

$$r(t) = [(2 + \cos(2t))\cos(3t), (2 + \cos(2t))\sin(3t), \sin(4t)]^T$$

parametrized on $[0, 2\pi]$. We talk more about this in class like that it lives on a torus and why you can not tie knots in $\mathbb{R}^n$ for $n > 3$. r is **simple** can be rephrased that the map $r : [0, 2\pi) \to \mathbb{R}^3$ is **injective**.

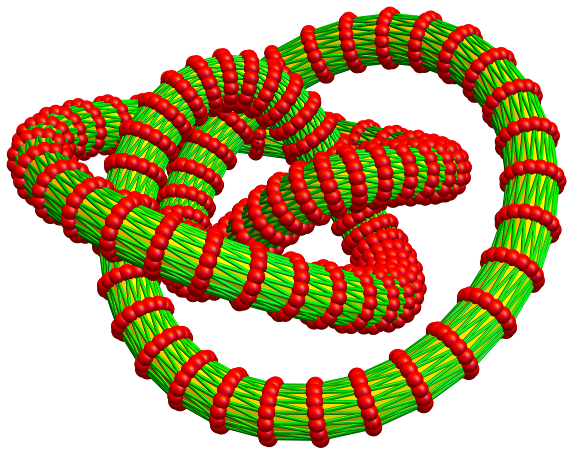

FIGURE 1. A picture of a knot. Drawing the curve in such a fancy way, needs concepts like curvature and torsion, which we will learn next week. The first fundamental form of the figure 8 knot is $r'(t)^2 = 101/2 + 36\cos(2t) + (5/2)\cos(4t) + 8\cos(8t)$. The action is $I = 101\pi$, the arc length involves elliptic integrals. Numerically it is $L = 42.966\ldots$. It is typical that we can explicitly give the action but not the length.

3.4. A curve is **parametrized by arc length** if $|\vec{r}\,'(t)| = 1$ for all t. You will prove in homework the following important result:

Theorem 2. *Every smooth regular curve in $\mathbb{R}^n$ can be parametrized by arc-length.*

3.5. It is custom to write $r(s)$ to indicate that we have an arc length parametrization. For the helix above, the arc length parametrization is $r(s) = [\cos(s/\sqrt{2}), \sin(s/\sqrt{2}, 1/\sqrt{2}]$. In general we do not bother to actually compute the arc length parametrization. Already in simple cases like the ellipse it would get nasty. We can use the theorem however to build theory and prove stuff about curves.

DIFFERENTIAL GEOMETRY

MATH 136

Unit 4: Frenet Equations

4.1. Today, we work at smooth curves $r : [a, b] \to \mathbb{R}^3$. [1] Define the **unit tangent vector** $T(t) = r'(t)/|r'(t)|$, the **normal vector** $B(t) = T'(t)/|T'(t)|$ and the **binormal vector** $B(t) = T(t) \times N(t)$. The three vectors are defined, as long as as r' and T' are non-zero. One calls it a **Frenet frame** (T, N, B). A smooth curve is called a **Frenet curve** if r', r'' are linearly independent at every t. This is equivalent to the statement $\boxed{r' \times r'' \neq 0}$ for every t.

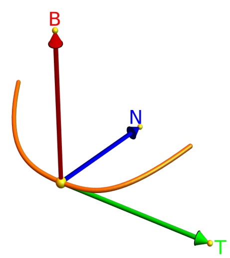

FIGURE 1. Direction T, normal N and binormal direction B.

Theorem 1. *For a Frenet curve, the Frenet frame TNB is orthonormal at every point.*

Proof. By assumption, r' and r'' are both not zero. If the parametrization is arc length, then $T = r'$ and $N = T'/|T'| = r''/|r''|$. Since $T \cdot T = 1$, we have by Leibniz product rule, $2T' \cdot T = 0$ so that N is perpendicular to T. The cross product $T \times B$ now also has length 1 and is perpendicular to both T and N. $\square$

4.2. Every Frenet curve $r(t)$ can be parametrized by arc length as you work out in the homework. The **curvature** κ is then defined as $\boxed{\kappa = T' \cdot N}$ which is $|T'|$. The curvature measures the deviation of the curve from being linear. The **torsion** τ is defined as $\boxed{\tau = N' \cdot B}$. It measures the deviation from the curve of being planar. We can encode the three vectors T, N, B by turning them into row vectors of an **orthogonal** 3×3 **matrix** $Q(t) = \begin{bmatrix} T & N & B \end{bmatrix}$. We get now $Q'(t) = K(t)Q(t)$, where $K(t)$ is skew-symmetric:

[1] In $\mathbb{R}^3$, one requires the map $r : [a, b] \to \mathbb{R}^3$ to be at least C^3.

4.3.

Theorem 2 (Frenet equations)**.** $\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$

Proof. Because $N \cdot T = 0$, we have $N' \cdot T = -T' \cdot N = -\kappa$. The relation $N \cdot N = 1$ implies $N' \cdot N = 0$. The relation $B \cdot B$ implies $B' \cdot B = 0$. Now expand with respect to the basis $\{T, N, B\}$ and use $\kappa = |T'|$ and $\tau = N' \cdot B$:

$$\begin{aligned} T' &= (T' \cdot T)T + (T' \cdot N)N + (T' \cdot B)B = 0 + 0 + \kappa N \\ N' &= (N' \cdot T)T + (N' \cdot N)N + (N' \cdot B)B = -\kappa T + 0 + \tau B \\ B' &= (B' \cdot T)T + (B' \cdot N)N + (B' \cdot B)B = 0 - \tau N + 0 . \end{aligned}$$

□

4.4. In the two dimensional case, we only have to consider T and N. We can reduce to the planar case if τ is constant 0. The Frenet equations can then be written as $\begin{bmatrix} T \\ N \end{bmatrix}' = \begin{bmatrix} 0 & \kappa \\ -\kappa & 0 \end{bmatrix} \begin{bmatrix} T \\ N \end{bmatrix}$.

4.5. The **fundamental theory of curves** in $\mathbb{R}^3$ tells that curvature and torsion determines a curve up to Euclidean congruences given by rotations or translations.

"The shape of a Frenet curve is determined by curvature and torsion".

Lemma 1. *For any smooth curvature and torsion functions $\kappa(t) > 0$ and $\tau(t)$, there exists up to translation and rotation a* **unique** *curve $r(t)$ parametrized by arc length that has the given curvature and torsion.*

Proof. Fixing an initial $r(0)$ and $(T(0), N(0), B(0))$ takes care of the translation and rotation part. The stage is set now to "build the curve". The functions $\kappa(t), \tau(t)$ define a **skew symmetric** matrix $K(t) = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}$. We look now for an orthogonal matrix $Q(t)$ such that the differential equation $Q'(t) = K(t)Q(t)$ holds. In the homework you verify that a differential equation $x' = F(t, x)$ with a C^1 function F has locally a unique solution and that if $x(t)$ stays bounded, the solutions exist for all times. You also check that if $Q(t)$ is a curve of orthogonal matrices, then $Q' = KQ$ with skew symmetric K. This can be reversed: if $Q(0)$ is orthogonal and $K(t)$ is skew symmetric, then the solution $Q(t)$ of the differential equation is orthogonal. Having now a solution $Q(t)$, it gives us $r'(s) = Q(s)r'(0)$. We have now $r'(t) = Q(s)r'(0)$. $r(t) = r(0) + \int_0^t r'(s)\, ds = r(0) + \int_0^t Q(s)r'(0)\, ds$. □

4.6. We have seen in the first warm-up class expressions for curvature and torsion for a curve $r(t)$. These formulas worked if the curve was not necessarily arc-length parametrized. In the Frenet case, meaning that $r' \times r'' \neq 0$, we will prove them in class:

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}, \quad \tau(t) = \frac{\det[r'(t), r''(t), r'''(t)]}{|r' \times r''|^2} .$$

Unit 5: Fundamental theorem of curves

5.1. A **Frenet curve** is given by a smooth map $[a,b] \subset \mathbb{R} \to \mathbb{R}^n$ for which $r', r'', \dots, r^{(n)}$ are linearly independent at every point. In the case $n = 3$ we had seen this as $r' \times r'' \neq 0$. Let $e_1, e_2, \dots, e_n$ denote the orthonormal frame obtained by Gram-Schmidt. One can get this as follows: build the matrix R with $r', r'' \dots, r^{(n)}$ as rows and perform the QR decomposition to get an orthonormal matrix Q in which the vectors $e_1, e_2, \dots, e_n$ are the rows. [1] Define the curvatures $\boxed{\kappa_j = e_j' \cdot e_{j+1}}$. It is positive for $j \leq n-2$. The largest κ_{n-1} is also called the **torsion** and is not necessarily positive. A natural generalization of the Frenet formulas to arbitrary dimensions is

Theorem 1 (Frenet-Serret formulas)**.**

$$\begin{bmatrix} e_1 \\ e_2 \\ \cdots \\ \cdots \\ \cdots \\ e_{n-1} \\ e_n \end{bmatrix}' = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 & \cdots & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 & \cdots & \cdots \\ 0 & -\kappa_2 & 0 & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 & \kappa_{n-1} \\ 0 & \cdots & \cdots & 0 & -\kappa_{n-1} & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \cdots \\ \cdots \\ \cdots \\ e_{n-1} \\ e_n \end{bmatrix} .$$

Proof. To get the entries of K expand e_j' in terms of the $e_1, \dots, e_n$.

$$e_j' = \sum_{i=1}^{n} (e_j' \cdot e_i) e_i \; .$$

This means $Q' = KQ$ with skew symmetric K. Especially, the diagonal entries of K are zero. The skew symmetry can be seen from $e_j \cdot e_k = 0$ for all $j \neq k$ implying $e_j' \cdot e_k = -e_j \cdot e_k'$. For every $j \leq n-1$, the e_j by definition are in the subspace generated by $r', r'', \dots, r^{(j)}$ which is the subspace generated by $e_1, \dots, e_j$ and e_j' therefore generated by $e_1, \dots, e_j$. This implies $e_j' \cdot e_{j+2} = e_j' \cdot e_{j+3} = \cdots = e_j' \cdot e_n = 0$. The only entry in the upper triangular part is $(e_j' \cdot e_{j+1}) = \kappa_j$. ∎

5.2. You verify the skew symmetry of K abstractly starting with $Q^T Q = 1$. A fancy way to restate is that in the Lie group $SO(n)$, the tangent space is the Lie algebra $so(n)$.

[1] Frenet and Serret have discovered the $n = 3$ dimensional case independently. The higher dimensional case has appeared only in the 20th century.

Lemma 1. *If $Q(t)$ is a curve of orthogonal matrices, then $Q' = AQ$ with skew symmetric A.*

5.3. Given curvatures $\kappa_1(t) > 0, \dots, \kappa_{n-2}(t) > 0, \kappa_{n-1}(t)$ which are all continuous, we get a continuous path $A(t)$ of skew symmetric matrices.

Theorem 2 (Fundamental theorem of curves)**.** *Given curvatures κ_j, there is up to translation and rotation a unique Frenet curve which has these curvatures.*

Proof. The curvatures define a curve $A(t)$ of skew symmetric matrices. The differential equation $Q' = A(t)Q = F(t, Q)$ is linear in Q and so smooth. Since the solution of this differential equation gives orthogonal matrices $Q(t)$ (check it!) the solution exists for all times. Proceed as in the 3 dimensional case by writing $r(t) = r(0) + \int_0^t r'(s)\, ds$ where $r'(s) = Q(s)r'(0)$ is given. □

5.4. Examples.
1) If K is constant, then e^{Kt} solves $Q' = KQ$.
2) If K is constant and $n = 3$, then the curve is a spiral if $\tau \neq 0$ and a circle if $\tau = 0$.
3) In $\mathbb{R}^3$, the torsion is constant zero if and only if the curve is contained in a plane.
4) In $\mathbb{R}^n$ the torsion is constant zero if and only if the curve is contained in a $(n-1)$ dimensional hyperplane.
5) A line is not a Frenet curve and the above does not apply.
6) For non-Frenet curves, lots of things can go wrong. Assume for example, you have a curve which contains some part which is a line. While traveling along that line, we can turn around and lose track of the Frenet frame.

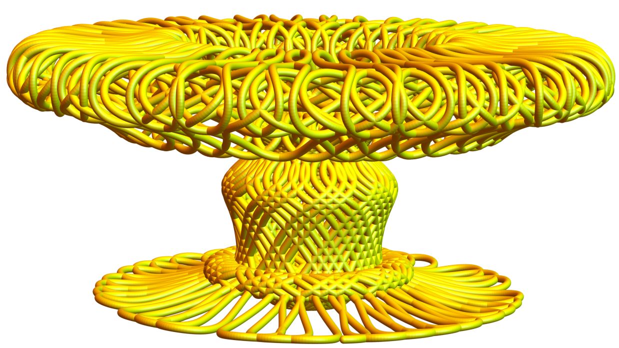

Figure 1. We see the unique curve with $\kappa(t) = 11 + 10\cos(17t), \tau(t) = 22\sin(5t)$ with $t \in [0, 65\pi]$. It is an entertaining fun to generate such curves.

5.5. A famous example is the Euler curve. It is a plane curve for which $\kappa(t) = t$ is fixed.

DIFFERENTIAL GEOMETRY

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Unit 6: Hopf Umlaufsatz

6.1. We look today at a theorem in two dimensions. It deals with **signed curvature** $\kappa = \frac{r' \times r''}{|r'|^3}$ using the cross product in 2 dimensions. [1]. We do not assume that the curve is Frenet. The curvature is allowed to become zero. We assume however that the curve is closed and regular meaning that $dr = r'$ is never zero. In that case, there is an arc length parametrization of the curve and $|\kappa| = |r''|$ because $r' \cdot r' = 1$ implies r'' is perpendicular to r'. But we have a signed curvature!

6.2. Assume that the curve is parametrized on $[a, b]$. The **rotation index** is defined as $\frac{1}{2\pi} \int_a^b \kappa(t)\, dt$. If the closed curve is not arc length parametrized, this is $\int_a^b \kappa(t)|r'(t)|\, dt$.

Theorem 1. *The rotation index of a closed C^2 curve is in $\mathbb{Z}$.*

Proof. Using arc length parametrization, write

$$r'(t) = [\cos(\alpha(t)), \sin(\alpha(t)]$$

then $\kappa = \alpha'$. Since the curve is closed, we have $\alpha(b) - \alpha(a) = 2\pi n$, where n is an integer. □

6.3. The case $r(t) = [\cos(nt), \sin(nt)]$ with $t \in [0, 2\pi]$ shows that the rotation index can take any integer value n. It is intuitively clear that if a curve has no self intersections, then the index must be either 1 or -1. This is not so obvious however. We do not want for example to refer to the Jordan curve theorem telling that a continuous simple closed curve in the plane divides the plane into an inside and outside. Heinz Hopf found a nice argument which proves this "Umlaufsatz" in an elegant way using a deformation picture:

Theorem 2 (Hopf Umlaufsatz)**.** *A simple closed regular C^2 curve has rotation index 1 or -1.*

Proof. Arc length parametrization is not needed. We assume that $r(t)$ is parametrized on the interval $[0, 1]$. Define on the square $Q = [0, 1] \times [0, 1]$ the function $f : Q \to \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})\}$ given by $f(t, s) = \arg((r(t) - r(s))/|r(t) - r(s)|)$ for $t \neq s$ and $\alpha(t) = f(t, t) = \arg(r'(t)/|r'(t)|)$ for $t = s$. Because $r \in C^1$, the function f is continuous. Now comes a homotopy argument. The index is $[f(1, 1) - f(0, 0)]/(2\pi)$ and is an integer. If we move along the diagonal and look at $\alpha(t) = f(t, t)$ we see a continuous curve which

[1]The cross product in n dimensions has $\binom{n}{2} = n(n-1)/2$ components. For $n = 2$ it is a scalar

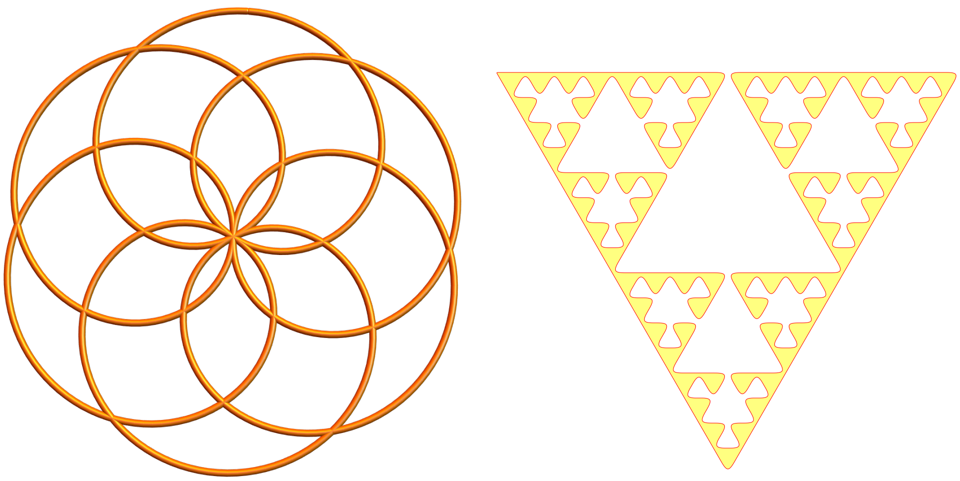

FIGURE 1. To the left the curve $r(t) = [\cos(t) + \cos(7t), \sin(t) + \sin(7t)], t \in [0, 2\pi]$ reminds of the flower of life. Its rotation number is 7. We can compute $\kappa(t)|r'(t)| = 4 + 72/(25 + 8\cos(6t))$ which integrates on $[0, 2\pi]$ up to 14π. To the right, a simple closed smooth curve in the plane. What is its rotation number?

moves on the circle $\mathbb{T}$. If we deform the curve the total change remains the same. We can continuously deform the curve so that we first deform from $(0, 0)$ straight to $(0, 1)$ and then straight from $(0, 1)$ to $(1, 1)$. Choose a coordinate system so that is in $y \geq 0$ just touching the x-axes. If $r'(0) = [a, 0]$ with positive a then $f(t, s) \in [0, \pi]$ with $f(0, 0) = 0$ and $f(0, 1) = \pi$ and then $f(1, 1) = 2\pi$. If $a < 0$, then $f(t, s) \in [-\pi, 0]$ with $f(0, 0) = \pi$ and $f(0, 1) = 0$ and then $f(1, 1) = -\pi$. In the former case, $i = 1$ in the later $i = -1$. □

6.4. Remarks:
1) This is a Gauss-Bonnet type result for a 2 dimensional flat manifold with boundary.
2) The proof shows that this even works for C^1 curves as $f(t, t) - f(s, s)$ is just the angle change of the tangent. This works even if the curvature is not defined. In the homework you even push it to polygons. Most texts assume C^2.

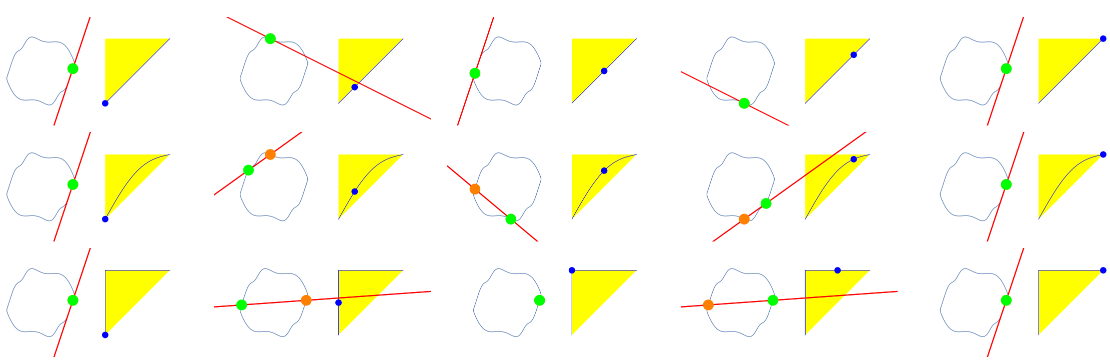

FIGURE 2. The deformation argument.

DIFFERENTIAL GEOMETRY

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Unit 7: Four vertex theorem

7.1. A **vertex** is a local maximum or minimum of the curvature function $\kappa(t)$. If you look at the case of an ellipse that is not a circle, you see two maxima and two minima. There are therefore 4 vertices. What happens in general? A curve is called **convex** if it bounds a convex region R. A region R is called **convex** if the line segment between any two points $A, B \in R$ is part of the region.

Theorem 1. *A simple closed regular convex C^3 plane curve has at least 4 vertices.*

Proof. We can assume κ is not constant. As a continuous function on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ it by the extremal value theorem at least one maximum a and one minimum b. Assume, $r(a)$ and $r(b)$ were the only critical points and that $[0, L]$ is the parameter interval. Choose the coordinate system so that these two points $r(a), r(b)$ are on the x-axis so that $r(t) = [x(t), y(t)]$ and $y(a) = y(b) = 0$. Convexity assures that there is no other root $y(t) = 0$. So, k' changes sign at $s = a$ and $s = b$ and nowhere else and $k'(s)y(s)$ does not change sign at all. The Frenet equations tell $e_1 = T = [x', y'], e_2 = N = [-y', x'], [x'', y''] = e_1' = \kappa e_2 = \kappa[-y', x']$ from which follows that $x'' = -\kappa y'$. Integration by parts gives $\int_a^{a+L} \kappa'(s)y(s)\, ds = \kappa y|_0^L - \int_a^{a+L} \kappa(s)y'(s)\, ds = \int_a^{a+L} x''(s)\, ds = x'(a+L) - x'(a) = 0$ which is not possible given that $\kappa'(s)y(s)$ does not change sign. There is therefore an other maximum or minimum. The number of local max and local min are the same for a periodic function (they must alternate as two successive maxima have a minimum between) so that there must be 4 vertices. $\square$

7.2. Examples.
1) The ellipse $r(t) = [2\cos(t), \sin(t)]$ has the curvature $\kappa(t) = 2(\cos^2(t) + 4\sin^2(t))^{-3/2}$ which has maxima at $t = 0, \pi$ and minima at $t = \pm\pi/2$.
2) The curve $r(t) = 5[\cos(t), \sin(t)] + [\cos(2t), \sin(2t)]$ has curvature $r'(t) \times r''(t)/|r'(t)|^{3/2}$ that has minima at $0, \pi$ and maxima at $\pm \arccos(-2/5)$.
3) The **limacon** $r(t) = [\cos(t), \sin(t)] + [\cos(2t), \sin(2t)]$ has curvature with only 2 vertices! Why is this not a counter example?

7.3. Remarks.
1) Convexity is not really needed. Osserman showed that for any simple closed C^2 curve C there are $2n$ components if the smallest circle enclosing C intersects it in at least n connected components. A general simple closed C^2 curve has at least 4 vertices.
2) V. Arnold conjectured that the result holds for any curve that can be obtained from a circle by suitable deformations.

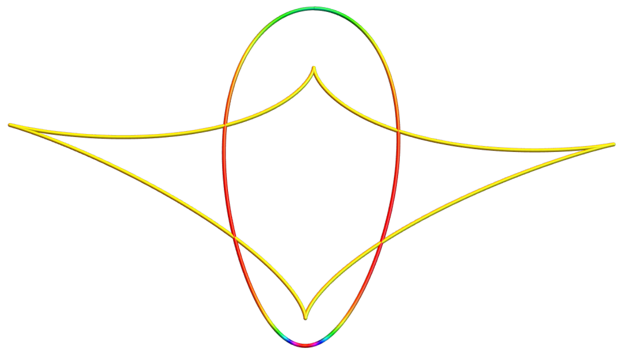

FIGURE 1. A simple closed curve. Color encodes curvature. The 4 vertices are visualized by plotting the **evolute** $e(t) = r(t) + n(t)/\kappa(t)$, where $n(t)$ is the normal vector pointing inside. The vertices of the curve correspond to cusps of the evolute.

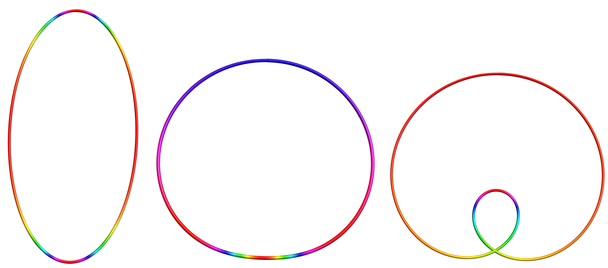

FIGURE 2. The ellipse, a deformation of an ellipse and a limacon.

3) If a continuous real valued, periodic function has at least two local maxima and two local minima, it is the curvature function of a simple closed curve.

7.4. History.

1) The theorem was proven in 1909 by Syamadas Mukhopadhyaya for convex curves. 2) The general case was published in 1912 by Adolph Kneser. 3) The proof given above in the convex case is due to G. Herglotz in 1930. 4) Robert Osserman in 1985 generalized the result to the "four or more vertex theorem". 5) The converse result started with H. Gluck in 1971 and was proven in 1997 by Bjoern Dahlberg.

7.5. Related.

1) The **evolute** of a plane curve is defined as $e(t) = r(t) + n(t)/\kappa(t)$. It is the caustic of the normal map. At points where $\kappa'(t)$ is zero, the evolute has cusps. A caustic of a smple closed curve therefore has at least 4 cusps. For the ellipse the evolute is called the Lamé curve.
2) The **tennis ball theorem** states that a C^2 curve on the sphere that divides the sphere into regions of equal area have at least 4 inflection points.
3) The open **last geometrical problem of Jacobi** asks whether a caustic on an ellipse has at least 4 cusps.

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DIFFERENTIAL GEOMETRY

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Unit 8: Fundamental Forms

8.1. A surface M in $\mathbb{R}^3$ is defined by a C^2 map $r : R \to r(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$ from a planar domain R to space $\mathbb{R}^3$. The partial derivatives r_u, r_v are tangent to the **grid curves** $u \to r(u,v)$ and $v \to r(u,v)$ and so tangent to M. If r is regular, the **unit normal vector** $n = r_u \times r_v / |r_u \times r_v|$ is defined and perpendicular to the surface.

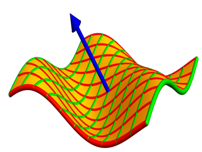

FIGURE 1. A parametrized surface $r(u,v)$ with unit normal vector $n(u,v)$. When seen as a map from M to S^2 it is known as the Gauss map.

8.2. We have already seen the **first fundamental form** $I = g = dr^T dr$ satisfy $\det(I) = |r_u \times r_v|^2$.

Theorem: First fundamental form:

$$I = g = \begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} r_u \cdot r_u & r_u \cdot r_v \\ r_v \cdot r_u & r_v \cdot r_v \end{bmatrix}$$

is a symmetric positive definite bilinear form.

Proof. The matrix dr is a 3×2 matrix and dr^T is a 2×3 matrix: $dr^T = \begin{bmatrix} - & r_u & - \\ - & r_v & - \end{bmatrix}$, $dr = \begin{bmatrix} | & | \\ r_u & r_v \\ | & | \end{bmatrix}$. The product is a 2×2 matrix. Now $g = dr^T dr$. We have seen already that $\det(g) = |r_u \times r_v|^2$. The trace of g is $\text{tr}(g) = |r_u|^2 + |r_v|^2$. Having positive trace and positive determinant assures that we have a positive definite matrix. We call g a **bilinear form** because it maps two vectors X, Y to a number $\langle X, Y \rangle = X^T g Y$. It defines us a scalar product on the surface. ∎

8.3. Examples:

1) In the case of a graph of a function $r(u,v) = \begin{bmatrix} u \\ v \\ f(u,v) \end{bmatrix}$ we have $g = \begin{bmatrix} 1+f_u^2 & f_u f_v \\ f_u f_v & 1+f_v^2 \end{bmatrix}$.

2) In the sphere case $r(u,v) = \begin{bmatrix} \sin(v)\cos(u) \\ \sin(v)\sin(u) \\ \cos(v) \end{bmatrix}$ we have $g = \begin{bmatrix} \sin^2(v) & 0 \\ 0 & 1 \end{bmatrix}$. Note that at $v=0$ and $v=\pi$ this is not regular.

8.4. If $r: R \to \mathbb{R}^3$ is a regular C^2 parametrization of a surface M, define

$$n(u,v) = \frac{r_u \times r_v}{|r_u \times r_v|} .$$

It is continuously differentiable because r was assumed to be C^2. The Jacobean derivative dn is the 3×2 matrix $dn = \begin{bmatrix} | & | \\ n_u & n_v \\ | & | \end{bmatrix}$. We can combine it with dr^T and define the **second fundamental form** $h = -dr^T dn$. It agrees with $(d^2 r)^T n$.

Theorem: Second fundamental form

$$II = h = \begin{bmatrix} L & M \\ M & N \end{bmatrix} = \begin{bmatrix} -r_u \cdot n_u & -r_u \cdot n_v \\ -r_v \cdot n_u & -r_v \cdot n_v \end{bmatrix} = \begin{bmatrix} n \cdot r_{uu} & n \cdot r_{uv} \\ n \cdot r_{vu} & n \cdot r_{vv} \end{bmatrix}$$

is a symmetric bilinear form.

Proof. From $r_u \cdot n = 0$ we get $r_{uu} \cdot n = -r_u \cdot n_u$ and similarly get $r_{uv} \cdot n = -r_u \cdot n_v$. Now, $II = -dr^T dn$ is symmetric because Clairaut applies. Clear is $r_u \cdot n_v = n_v \cdot r_u$. □

8.5. The **third fundamental form** is $III = e = dn^T dn$ is the first fundamental form of the sphere map n.

Theorem: Third fundamental form:

$$III = e = \begin{bmatrix} n_u \cdot n_u & n_u \cdot n_v \\ n_v \cdot n_u & n_v \cdot n_v \end{bmatrix} .$$

is a symmetric bilinear form and $|n_u \times n_v|^2 = \det(III)$.

Proof. $III = dn^T dn$ is symmetric as the dot product is commutative. The proof of $|n_u \times n_v|^2 = \det(III)$ is word by word identical what we have done in the second class for $r_u \times r_v$. □

8.6. The third fundamental form is not independent from the other two fundamental forms. In homework: with $H = \text{tr}(A)/2$ and $K = \det(A)$ are trace and determinant of $A = I^{-1} II$:

Theorem: Compatibility: $III - 2H II + K I = 0$

DIFFERENTIAL GEOMETRY

MATH 136

Unit 9: Curvature

9.1. The **shape operator** S or **Weingarten map** encodes how the surface curves in $\mathbb{R}^3$. S maps the tangent space T_pM to the tangent space T_pS^2. It maps r_u to $-n_u$ and r_v to $-n_v$, meaning $Sdr^T = -dn^T$. By identifying T_pM with T_pS^2 it is a self-map of T_pM. In the basis $\{r_u, r_v\}$, it becomes a 2×2 matrix $A = S^T$ satisfying $\boxed{dn = -drA}$. Take this matrix equation for 3×2 matrices as the relation defining the shape operator.[1]

Theorem 1 (Shape operator). *The shape operator matrix is* $\boxed{A = I^{-1}II}$.

Proof. Using the 3×2 matrices dr, dn we have defined A as $dn = -drA$. The second fundamental form is $II = -dr^T dn = dr^T drA = IA$. Since I is invertible, we can solve for A and get $A = I^{-1}II$. □

While A is not necessarily symmetric, it is symmetric with respect to the inner product $\langle v, w \rangle = v^T gw$. Proof $\langle Av, w \rangle = (Av)^T Iw = v^T A^T I = v^T II^T (I^{-1})^T Iw = v^T II^T w = v^T IIw = w^T IIv$ because II was symmetric. Having been able to switch v,w shows $\langle Aw, v \rangle = \langle w, Av \rangle$.

9.2. Define the **Gaussian curvature** as $\boxed{K = \det(A)}$. Written out, the curvature is

$$K = \det(A) = \frac{\det(II)}{\det(I)} = \frac{LN - M^2}{EG - F^2} = \lambda\mu \ .$$

From linear algebra, we know it is the product of the eigenvalues λ, μ of A. The **mean curvature** H is defined as the average of eigenvalues λ, μ of A. It is

$$H = \frac{\mathrm{tr}(A)}{2} = \frac{EN - 2FM + GL}{2(EG - F^2)} = \frac{\lambda + \mu}{2} \ .$$

Theorem 2.

$$K = \frac{\det(II)}{\det(I)}$$

is independent of the basis.

Proof. We immediately have from the product determinant formula and the fact that I is invertible that $\det(K) = \det(A) = \det(II)/\det(I)$. Since determinants are independent of the basis, also the curvature is. □

[1]Einstein: $I = g_{ij}$ and $I^{-1} = g^{ij}$ and $II = h_{ij}$ and $A^k_i = g^{kj}h_{ji}$. The shape operator is a "linear transformation" $A^i_k v^k = w^i$ on vectors. I, II are quadratic forms **"(0,2) tensor fields"** while A is a transformation at every point, a **"(1,1) tensor field"**. $dn = -drA$ are called Weingarten equations.

9.3. We write $\iint_M f dV$ for the integral $\iint_R f(u,v)|r_u \times r_v| dudv$. For $f = 1$, this is the **surface area** $|M| = \iint_R |r_u \times r_v| \; dudv$. Since $n(u,v)$ parametrizes the unit sphere, we have $\iint_R |n_u \times n_v| \; dudv = 4\pi$. For convex surfaces, we can use the same parameter domain $R = [0, 2\pi) \times [0, \pi)$ and see that the total curvature is the same than the total curvature of a sphere. This requires that K is positive. The area of the image of S is called the **total curvature**. We have now already a cool version of Gauss-Bonnet: The general version will work for any surface, not only for convex (and so positive curvature) surfaces.

Theorem 3 (Gauss-Bonnet for convex closed surfaces)**.** *$\iint_M K dV = 4\pi$.*

Lemma 1. $\boxed{III = IIA}$ *and so* $\det(III) = \det(A)^2 \det(I)$.

Proof. Start with the definition $dn = -drA$. Multiply with dn^T from the left to get $III = dn^T dn = -dn^T drA = IIA$. Taking determinants gives $\det(III) = \det(II)\det(A) = \det(I)\det(A)\det(A)$. □

9.4. The two identities $II = IA$ and $III = IIA$ can be used for a proof of the identity $\boxed{III - 2HII + KI = 0}$ without using the inner product defined by I. [2] Now to the proof of the Gauss-Bonnet result:

Proof. Take square roots of the lemma gives $\sqrt{\det(III)} = K\sqrt{\det(I)}$. This step has required K to be non-negative. Therefore,

$$\begin{aligned} 4\pi &= \iint_R |n_u \times n_v| dudv = \iint_R \sqrt{\det(III)} \; dudv \\ &= \iint_R K\sqrt{\det(I)} \; dudv = \iint_R K|r_u \times r_v| dudv = \iint K \; dV \; . \end{aligned}$$

□

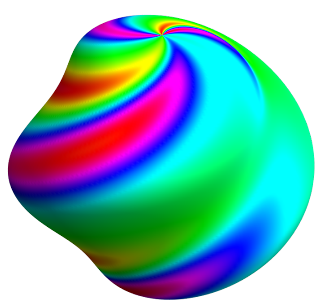

FIGURE 1. We see a convex surface colored with the curvature function K. Gauss-Bonnet establishes that the total curvature is 4π.

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[2]Thanks to some students of the course to point this out.

DIFFERENTIAL GEOMETRY

MATH 136

Unit 10: Discrete Gauss-Bonnet

10.1. To see the Gauss-Bonnet result for a general surface M, we need to define **Euler characteristic** $\chi(M)$. It is defined by a **triangulation** of M. To define equivalence of triangulations, we use the language **graph theory**, initiated by Leonhard Euler.

10.2. A **graph** $G = (V, E)$ is a finite set V of **vertices** or **nodes** and a finite set E of different **edges** or **connections** (a, b) with $a \neq b$. Every subset V' of V **generates** a subgraph (V', E'), where $E' = \{(a, b) \in E, a \in V', b \in V'\}$. We can so associate a subset V' of V the subgraph it generates. A pair of adjacent vertices for example generates a K_2 subgraph. A pair of non-adjacent vertices generates $S^0 = \overline{K_2}$ the graph with two points and no vertices, which is also known as the **0-sphere**.

10.3. A circular graph C_n with $n \geq 4$ vertices is called a **circle** of length n. The **unit sphere** $S(v)$ of a vertex v is the subgraph generated by all immediate neighbors of v. A **2-manifold** is a graph for which every unit sphere is a circle. A 2-manifold graph G embedded as a subset $|G| \subset M$ defines a **triangulation** of M; $v \in V$ is realized as a point in M, an edge $e \in E$ is realized as a simple curve in M parametrized by an interval, a connected component in the complement of $|G|$ is regularly parametrized by a triangle $R \subset \mathbb{R}^2$.

10.4. A complete subgraph K_3 of G is also called a **triangle** or a **face** in G. The **Euler characteristic** of a 2-manifold is defined as $\chi(G) = |V| - |E| + |F|$, where $|X|$ is the **cardinality** of X. The **curvature** of a 2-manifold is defined as $\boxed{K(v) = 1 - |S(v)|/6}$. The following theorem goes back to **Victor Eberhard**.

Theorem 1 (Gauss-Bonnet). *For a 2-manifold,* $\sum_{v \in V} K(v) = \chi(G)$.

Proof. Define the function $\omega(x)$ on $X = V \cup E \cup F$ as $\omega(x) = (-1)^{\dim(x)}$ where $\dim(x) = |x| - 1$ is the dimension one less than the number $|x|$ of vertices in x. So, $\chi(G) = |V| - |E| + |F| = \sum_{|x|=1} (-1)^0 + \sum_{|x|=2} (-1)^1 + \sum_{|x|=3} (-1)^2 = \sum_{x \in X} \omega(x)$. If all values -1 from an edge (a, b) are distributed equally to (a, b) and all the values 1 from a face (a, b, c) are distributed equally to the vertices a, b, c, we end up with a function K that is only non-zero on vertices v and equal there to $K(v) = 1 - S_0(v)/2 + S_1(v)/3$, where $S_0(v), S_1(v)$ are the number of vertices and edges in $S(v)$ for $v \in V$. In the case of a circular $S(v)$ we know $S_0(v) = S_1(v) = |S(v)|$ so that $K(v) = 1 - |S(v)|(1/3 - 1/2) = 1 - |S(v)|/6$. $\square$

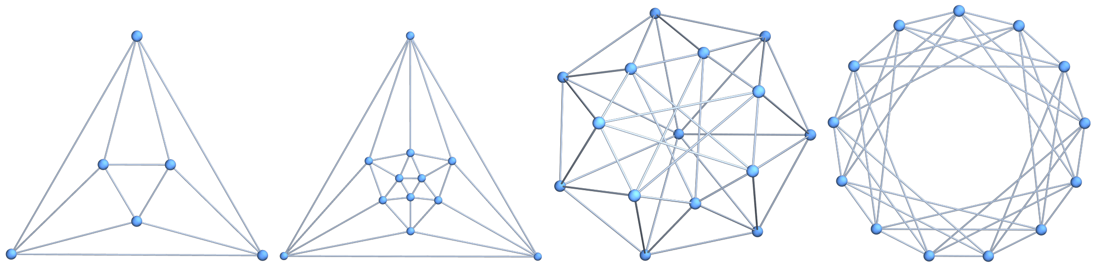

FIGURE 1. The **octahedron** has $K(v) = 1/3$ for all v. The **icosahedron** has $K(v) = 1/6$ for all v. A **projective plane** has curvatures in $\{0, 1/6, -1/6\}$. The flat torus = Clifford torus has constant 0 curvature.

10.5. An **edge collapse** $G \to G'$ takes $(a,b) \in E$ and identifies a with b. It removes 2 faces, 3 edges and 1 vertex so that $\chi(G) = \chi(G')$. A 2-manifold G is a **2-sphere** S^2, if $\chi(G) = 2$. The **connected sum** $G\#H$ of two 2-manifolds G, H is obtained by removing an edge in both manifolds and identifying the C_4 boundaries of the holes. If $v \in V(G)$ and $w \in V(H)$, with $|S(v)| = |S(w)|$, one can also remove v from G and w from H and glue boundaries to get a $G\#H$ with $\chi(G\#H) = \chi(G) + \chi(H) - 2$. A **2-ball** is a graph obtained from a S^2 by removing a vertex v. A **2-cylinder** or **handle** is a 2-sphere in which two vertices in distance > 2 removed. A **2-torus** is a 2-manifold obtained from a 2-cylinder by gluing the boundaries, matching orientation. A **Moebius strip** is a projective plane with one vertex removed. When glued into a hole of a sphere it is a **cap**. The **Klein bottle** is a S^2 with two caps. The projective plane is a sphere with a cap. The **boundary** of a G is $\{w \in V | S(v)$ is not a circle $\}$. The boundary of a ball or a Moebius strip is a circle. 2-manifolds have no boundary.

10.6. A **topological deformation** of a 2-manifold G takes a 2-ball in G and replaces it with an other 2-ball with the same boundary. In other words, a topological deformation is the process $G \to G' = G\#S^2$ implying $\chi(G) = \chi(G')$. Two 2-manifolds G, H are **topologically equivalent** if they can be deformed into each other by a finite set of topological deformations. An example of a topological deformation is to take out an edge and fill in the opposite diagonal edge. This **diagonal flip** is known as **Pachner transformation**. The following theorem is a milestone of 19'th century mathematics:

Theorem 2 (Classification of 2-manifolds)**.** *Every connected 2-manifold is equivalent to a 2-sphere S^2 or a connected g-sum of either $\mathbb{T}^2\#\cdots\#\mathbb{T}^2$ or $\mathbb{P}^2\#\cdots\#\mathbb{P}^2$.*

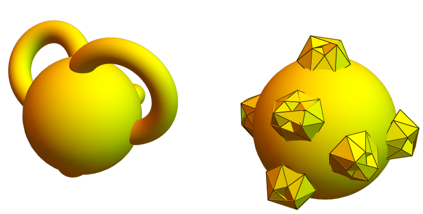

FIGURE 2. A 2-manifold is either S^2 with g handles (orientable) and $\chi = 2 - 2g$ or a S^2 with g cross caps and $\chi = 2 - g$ (non-orientable).

DIFFERENTIAL GEOMETRY

MATH 136

Unit 11: Geodesics

11.1. If $M = r(R)$ is a regular manifold, define the space X of regular paths $x(t)$ that start at $x(a) \in R$ and end at $x(b) \in R$. If $F(x, \dot{x})$ is a function of position x and velocity $\dot{x}$, we can minimize $E(x) = \int_a^b F(x, \dot{x})\ dt$ by looking for paths $x(t)$ at which the variation is zero. [1]

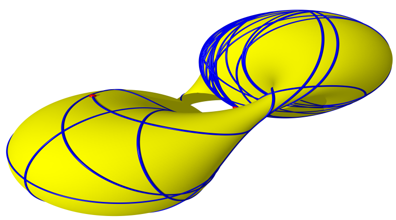

FIGURE 1. This geodesic on a torus was computed in Povray with the Schield's ladder method: evolve freely in $\mathbb{R}^3$ but stay glued to the surface.

Theorem 1 (Euler-Lagrange). *If x minimizes E, then* $\boxed{F_x(x, \dot{x}) = \frac{d}{dt} F_{\dot{x}}(x, \dot{x})}$.

Proof. For a minimum, the change $E(x + \xi) - E(x)$ of a displacement $x + \xi$ of x satisfies $\int_a^b F(x + \xi, \dot{x} + \dot{\xi}) - F(x, \dot{x})\ dt \geq 0$. As Fermat knew, we better have $dE\xi = \lim_{h\to 0}(E(x+h\xi) - E(x))/h = 0$ because a non-zero limit would make $E(x+h\xi)$ larger or smaller than $E(x)$ for small h. By the chain rule, $dE\xi = \int_a^b F_x(x, \dot{x})\xi + F_{\dot{x}}(x, \dot{x})\dot{\xi}\ dt$. Integration by parts, using $\xi(a) = \xi(b) = 0$, gives $dE\xi = \int_a^b [F_x(x, \dot{x}) - \frac{d}{dt} F_{\dot{x}}(x, \dot{x}]\xi(t)\ dt$. In order that this is zero for all ξ, we better have $[F_x(x, \dot{x}) - \frac{d}{dt} F_{\dot{x}}(x, \dot{x})] = 0$ for all $t \in [a, b]$. Proof. If $\neq 0$ at some point $t \in [a, b]$, it would be non-zero in a neighborhood U of t, allowing to find a smooth function ξ that is positive in U and 0 else, producing a nonzero change $dE\xi$. ☐

11.2. To understand minima if $F(x, \dot{x}) = \langle \dot{x}, \dot{x} \rangle = g(x)(\dot{x}, \dot{x}) = \sum_{i,j} g_{ij}(x)\dot{x}^i\dot{x}^j$, we need notation. M was defined as a regular map $r : R \to \mathbb{R}^n$ giving points $r(u^1, \dots, u^m) \in \mathbb{R}^n$. Define the **Christoffel symbols** $\boxed{\Gamma_{ijk} = r_{u^i u^j} \cdot r_{u^k}}$. The product rule gives

$$\partial_{u^k} g_{ij} = r_{u^i u^k} \cdot r_{u^j} + r_{u^i} \cdot r_{u^j u^k} = \Gamma_{ikj} + \Gamma_{jki}\ ,$$

[1] $x(t) = (x^1(t), \dots, x^m(t)) = (u^1(t), \dots u^m(t))$ as most texts use this notation. For typographical reasons, write $\dot{x}^k$ rather than x'^k. If $r(u, v)$ parametrizes M, paths $x(t) = (u(t), v(t)) \in R$ define curves $r(x(t)) \in M$. "Variation" instead of "derivative" avoids confusion with $\dot{x}$. Variations are directional derivatives in an infinite dimensional space X of paths between two fixed points.

$$\partial_{u^i} g_{jk} = r_{u^j u^i} \cdot r_{u^k} + r_{u^j} \cdot r_{u^k u^i} = \Gamma_{jik} + \Gamma_{kij} \ ,$$
$$\partial_{u^j} g_{ki} = r_{u^k u^j} \cdot r_{u^i} + r_{u^k} \cdot r_{u^i u^j} = \Gamma_{kji} + \Gamma_{ijk} \ .$$

Adding the second and third and subtracting the first, using Clairaut $\Gamma_{ijk} = \Gamma_{jik}$, gives $2\Gamma_{ijk}$ on the right hand side. So:

Lemma 1. $\boxed{\Gamma_{ijk} = \frac{1}{2}[\frac{\partial}{\partial u^i} g_{jk} + \frac{\partial}{\partial u^j} g_{ki} - \frac{\partial}{\partial u^k} g_{ij}] \ .}$

Using the notation $g^{ij} = (g^{-1})_{ij}$ and $\Gamma^k_{ij} = \sum_{l=1}^m g^{kl}\Gamma_{ijl}$, we get to the main point: [2]

Theorem 2 (Geodesics)**.** *Minima of the action functional $E(x) = \int_a^b \langle \dot{x}, \dot{x} \rangle \ dt$ satisfy*

$$\boxed{\ddot{x}^k + \sum_{i,j=1}^m \Gamma^k_{ij} \dot{x}^i \dot{x}^j = 0.}$$

Proof. To show that Euler-Lagrange for $F(x, \dot{x}) = \sum_{i,j} g_{ij}(x)\dot{x}^i\dot{x}^j$ is $2\sum_j g_{jk}\ddot{x}^j + 2\sum_{i,j} \Gamma_{ijk}\dot{x}^i\dot{x}^j = 0$: use notation $\partial_{x^k} g_{ij} = g_{ij,k}$. First get $F_{\dot{x}^k} = \sum_j g_{kj}\dot{x}^j + \sum_j g_{jk}\dot{x}^j$. Then $\frac{d}{dt}F_{\dot{x}^k} - F_{x^k} = \sum_{j,i} g_{kj,i}\dot{x}^i\dot{x}^j + \sum_j g_{kj}\ddot{x}^j + \sum_{j,i} g_{jk,i}\dot{x}^i\dot{x}^j + \sum_j g_{jk}\ddot{x}^j - \sum_{i,j} g_{ij,k}\dot{x}^i\dot{x}^j$. The 1st, 3rd and 5rd terms add up to $2\sum_{i,j} \Gamma_{ijk}\dot{x}^i\dot{x}^j$. The 2nd and 4th give $2\sum_j g_{jk}\ddot{x}^j$. □

We see that the acceleration of a particle moving on a geodesic is determined by the velocity and "gravitational force" terms Γ which involves changes in the metric. Einstein would interpret these changes in metric as "mass".

11.3. With $G(x, \dot{x}) = \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j} = \sqrt{2F}$, we get the **arc length functional**

$$I(r) = \int_a^b ||\dot{x}|| \ dt = \int_a^b \sqrt{\langle \dot{x}(t), \dot{x}(t) \rangle} \ dt = \int_a^b \sqrt{\sum_{i,j} g_{ij}(x)\dot{x}^i\dot{x}^j} \ dt \ .$$

Theorem 3 (Maupertius)**.** *Action and length functionals have the same extrema.*

Proof. Using the chain rule, the Euler-Lagrange equations $\frac{d}{dt}G_{\dot{x}} = G_x$ are $\frac{d}{dt}F_{\dot{x}}/\sqrt{2F} = F_x/\sqrt{2F}$. Because x was regular, $F(x, \dot{x})$ is never zero and the Euler-Lagrange equations of F and G are equivalent. □

11.4. If $x(t)$ is an arc length parametrized curve on M, the **normal curvature** is defined as $\boxed{\kappa_n = \ddot{x} \cdot n}$. It is the scalar projection acceleration $\ddot{x}$ onto n. It is smaller or equal than $\kappa = |\ddot{x}|$. Define the **geodesic curvature** as $\kappa_g = (\dot{x} \times n) \cdot \ddot{x}$. Pythagoras gives $\kappa_n^2 + \kappa_g^2 = \kappa^2$. Note that both κ_n and κ_g can be signed.

Theorem 4 (Schield's ladder)**.** *Geodesics have zero geodesic curvature.*

Proof. Geodesic curvature is $||\ddot{x}|| = \sqrt{\langle \ddot{x}, \ddot{x} \rangle}$. If positive, there would be an acceleration tangent to the surface and so a shorter connection between $x(t)$ and $x(t + 2h)$ than $x(t), x(t + h), x(t + 2h)$. Think like Archimedes: geodesics have no intrinsic curvature. □

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[2] It's musical! ∂_{u^i} is co-variant and u^i is contra-variant. Einstein would write $g_{ij}\dot{x}^i\dot{x}^j$ for $\langle \dot{x}, \dot{x} \rangle$.

DIFFERENTIAL GEOMETRY

MATH 136

Unit 12: The exponential map

12.1. The geodesic differential equation $\boxed{\ddot{x}^k + \sum_{i,j} \Gamma^k_{ij} \dot{x}^i \dot{x}^j = 0}$ can be written as a first order system $\frac{d}{dt}[x, \dot{x}] = [\dot{x}, f(x, \dot{x})]$ if the first fundamental form g is twice differentiable. This ordinary differential equation therefore has **local solutions** for some time $t \in (-a, a)$ by the Piccard existence theorem. But the solutions exist for all time. No "blow up" is possible if the surface is smooth, regular and closed. The reason is the following:

Lemma 1. *If $x(t)$ is geodesic, then $\langle \dot{x}, \dot{x} \rangle = \sum_{i,j} g_{ij} \dot{x}^i \dot{x}^j$ is preserved.*

Proof. Either note $\frac{d}{dt}\langle \dot{x}, \dot{x} \rangle = 2\langle \ddot{x}, \dot{x} \rangle = 0$. Alternatively, define the Hamiltonian $H = -F + \sum_j \dot{x}^j F_{\dot{x}^j}$. Using the Euler-Lagrange equations, we get $\frac{d}{dt} H = -\sum_j F_{x^j} \dot{x}^j - \sum_j F_{\dot{x}^j} \ddot{x}^j + \sum_j \ddot{x}^j F_{\dot{x}^j} + \sum_j \dot{x}^j \frac{d}{dt} F_{\dot{x}^j} = 0$. (Now replace the last term with $\sum_j \dot{x}^j F_{x^j}$.) For $F(x, \dot{x}) = \langle \dot{x}, \dot{x} \rangle$, we have $H = -F + 2\sum_j \dot{x}^j \dot{x}^j = -F + 2F = F$. We see $H = F = \langle \dot{x}, \dot{x} \rangle$ is an energy that is preserved. □

Theorem 1 (Hopf-Rynov)**.** *For regular, compact, smooth M, geodesics exist globally.*

Proof. If M is C^4 then Γ is differentiable. The Piccard existence theorem gives local solutions in the unit tangent bundle $(p, v) \in M \times S^{m-1}$. A regular compact manifold is complete in the sense that all Cauchy sequences have limits. The only way that a solution path could not be continued is that $\dot{x}(t)$ blows up. Otherwise, we could restart the differential equation at the end point a of a maximal interval $(-a, a)$ of existence. By the lemma, a blow up of $\dot{x}(t)$ is not possible. □

12.2. Remarks: a) The regularity is necessary. On a piece-wise smooth manifold like a cube, a geodesic hitting a corner can not be continued continuously. **b)** There are compact Lorentzian manifolds like the Clifton-Pohl torus that are not complete. **c)** The lemma is important. The proof shows that each variational problem gets with a **Legendre transform** $H = -F + \sum_j \dot{x}^j F_{x^j}$ to an "energy" H that is preserved. [1]

12.3. The **exponential map** $\exp_p : T_pM \to M$ is obtained by defining $\exp_p(0) = p$ and for $v \neq 0$, define $\exp_p(v)$ by taking $v/|v|$ as initial direction of the geodesic flow and evolving it for time $|v|$. The image $\exp(S_r(0)) = W_r(p)$ is called the **wave front**. It is the set of all points which can be reached from p by running from it a geodesic of length p. Wave fronts are **geodesic circles** for small t but in general become very complicated.

[1] For more, see J. Moser, Selected Topics in the Calculus of Variations. (Notes by O. Knill) 2002

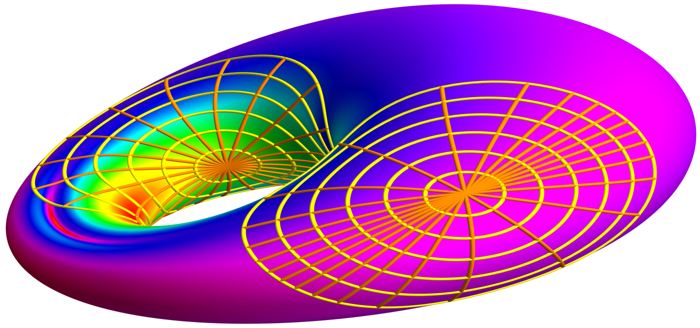

FIGURE 1. The exponential map evolves all possible geodesics from p. If all these geodesics are stopped at time t, we get a wave front $W_t(p)$.

12.4. The **radius of injectivity** of M is the smallest r such that the exponential map $B_r \subset T_pM \to M$ is injective.

Lemma 2. *For a compact manifold M, the radius of injectivity is positive.*

Proof. When fixing a point p, there is a $B_r(0) \subset \mathbb{R}^m$ such that that $\exp_p$ is invertible. This follows from the **inverse function theorem** and the fact that $d\exp_p = 1$ (identity matrix) at p because $\exp_p(v) - v = O(|v|^2)$ by definition. Let $r(p)$ be maximal radius on which $\exp_p(B_r(0))$ is differentiable. This function $r(p)$ is continuous in p and positive. By compactness of M and the **extremal value theorem**, there is a minimum, a lower bound. □

12.5. For fixed p, critical values of $\exp_p$ form the **caustic** of p. If r is the radius of injectivity, the open set $U = \exp_p(B_r(0)) \subset M$ is called the **normal neighborhood** of p. Lets look at the two dimensional case:

Lemma 3. *On U there are coordinates (ρ, θ) such that $g = I = \begin{bmatrix} 1 & 0 \\ 0 & G \end{bmatrix}$ satisfying $\lim_{\rho \to 0} G(\rho, \theta) = 1$.*

Proof. These are called **geodesic polar coordinates** because they come from the exponential map. Since velocity is preserved, the radial direction does not expand. □

12.6. This implies:

Theorem 2 (Gauss Lemma)**.** *For every unit vector v, the radial geodesics $\{\exp_p(sv), s \leq t\}$ is normal to the wave front $W_t(p)$.*

Proof. Within $U = \exp_p(B_r(0))$ this is clear by the coordinates. □

12.7. Remarks. 1) Geodesic coordinates with $I = g = \text{diag}(1, g_{22} \dots, g_{mm})$ exist on any m-manifolds. **2)** For 2-manifolds, linearising the geodesic flow affects only the vector perpendicular to the geodesic $x(t)$. This is called a **Jacobi field**. For surfaces, for fixed p and v, we get a **Jacobi differential equation** $\boxed{z'' = -K(x(t))z}$, where $z(t) = G(x(t))$ in the normal patch. The roots of $z(t)$ belong to caustic points $\exp_t(p)$.

Unit 13: Curvature is a Curl

13.1. The proof of the **Gauss-Bonnet theorem** will invoke **Green's theorem** from calculus. Also the **Theorema egregium** will boil down to the fact that curvature form KdV is the curl dX of a 1-form X, that only depends on the first fundamental form I. **Differential geometry** so builds heavily on **multi-variable calculus**.

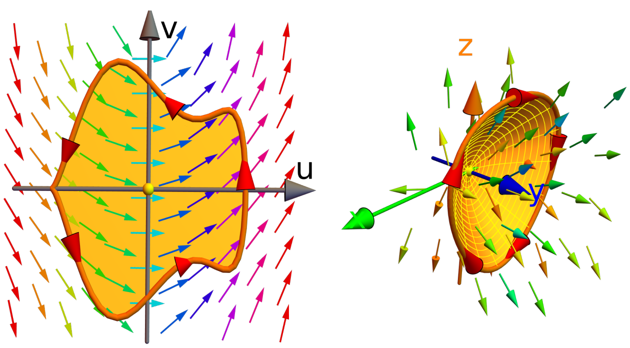

FIGURE 1. **Green's theorem** integrates the 2D curl dX over a planar region R. **Stokes theorem** integrates the 3D curl dF over a surface M. If $M = r(R)$, one can **pull back** the 1-form F in $\mathbb{R}^3$ to a 1-form X in $\mathbb{R}^2$ and so get Stokes from Green: $dF(r_u, r_v) = \text{curl}(F) \cdot r_u \times r_v = F_u \cdot r_v - F_v \cdot r_u = \text{curl}(X)$ for $X = [F \cdot r_u, F \cdot r_v]$ (see homework). In differential geometry, a particular X will lead to Gauss-Bonnet.

13.2. Green's theorem is usually written for planar vector fields $X^T = \begin{bmatrix} P \\ Q \end{bmatrix}$: the double integral of the curl dX of X in a R agrees with the line integral of X along the boundary δR. If we change to row vectors, we have a **1-form** $X = [P, Q]$. 'Power=force times velocity" $\begin{bmatrix} P \\ Q \end{bmatrix} \cdot \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix}$ is then the **matrix product** $X\dot{x}$ rather than $X^T \cdot \dot{x}$. [1]

13.3. Assume $X = [P, Q]$ is a 1-form and assume $x(t) = \begin{bmatrix} u(t), v(t) \end{bmatrix}^T$ is the parameterization of a **closed curve** δU with $t \in [0, L]$ bounding the region $U \subset R$. The **curl** of X is defined as $dX = \text{curl}(X) = Q_u - P_v$. The **1-form** X is a linear map which assigns to a contra-variant vector (column vector) like $\dot{x}$ a number $X\dot{x} = P\dot{u} + Q\dot{v}$.

[1] Both in physics as well in differential geometry, it is important to distinguish between **contra-variant objects** like vector fields $\nabla f = X^i$ and **co-variant objects** like 1-forms $df = X_i = \partial_{x^i} f$.

A 1-form especially can be integrated along a curve $\int_0^L X(x(t))\dot{x}dt$, the **line integral**. The curl of X is denoted by dX. It is a **2-form** which can be integrated over U. The **general Stokes theorem** tells $\boxed{\int_M dX = \int_{\delta M} X}$ if M is a k-manifold with boundary δM and X is a $(k-1)$-form then dX is a k-form. In the case $k = 2$, where X is a 1-form and $U \subset R$ is a region, we have

Theorem 1 (Green). $\int_{\delta U} X = \int_0^{2\pi} X(x(t))\dot{x}(t)\, dt = \iint_U \text{curl}(X)(u,v)\, dudv = \int_U dX.$

13.4. In calculus, you see this using vector fields $F = X^T$, meaning that every point is attached a contra-variant vector. In order to pair this with the velocity vector $\dot{x}$, we had to invoke the **dot product** $v \cdot w = v^T w$ and write a **matrix product** $X(x(t))\dot{x}(t)$. The just formulated version of Green's theorem is completely equivalent.

13.5. The key of Gauss Bonnet is to see that the curvature 2-form $K|r_u \times r_v|$ can be written as the curl dX of a 1-form X. Gauss-Bonnet theorem in the convex case is stated as $\iint_R K|r_u \times r_v|duv = 2\chi(M)$. A second computation will then show that if $M = r(U)$ is a manifold with boundary $r(x) = \delta(M)$, integrating the geodesic curvature along the boundary curve x is a **line integral** of X along x plus 2π. Gauss-Bonnet for surface patches $r(U)$ with boundary $t \to r(x(t))$ will then follow from Green's theorem.

13.6. Assume that $r : R \to \mathbb{R}^3$ is a regular parametrization of the surface M. A simple closed curve $x(t), t \in [0, L]$ encloses a region $U \subset R$ matching orientation. It defines a curve $r(x(t))$ bounding the manifold $r(U) \subset M$. We can assume that $x(t) = (u(t), v(t))$ is parametrized by arc length. At every point $p = r(u, v) \in M$, the vectors $\{r_u, r_v\}$ form a basis of the tangent space T_pM. Let $\{z, w\}$ be the Gram-Schmidt orthonormalized basis obtained from $\{r_u, r_v\}$ and the unit normal vector $n = r_u \times r_v/\sqrt{r_u \times r_v} = z \times w$.

13.7. The following lemma shows that we can attach two vectors z, w to every point p on the surface. It will allow us to define the 1-form $\boxed{X = zdw = [z \cdot w_u, z \cdot w_v]}$.

Lemma 1. *$z = ar_u, w = br_u + cr_v, n = z \times w$ form an orthonormal frame with functions a, b, c that only depend on the first fundamental form.*

Proof. Gram-Schmidt proceeds as follows $z = r_u/\sqrt{r_u \cdot r_u} = r_u/\sqrt{E} = ar_u$ and gets w as the normalization $br_u + cr_v$ of $r_v - (r_v \cdot z)z = r_v - (r_v \cdot r_u)r_u/E = r_v - \frac{F}{E}r_u$. $\square$

13.8. We will see next time that X can be computed from I alone and that

Lemma 2 (Curvature is a curl). *The curl satisfies* $\boxed{dX = Q_u - P_v = K\sqrt{\det(g)}}$.

13.9. For now, this is just an announcement. The computation comes next class. But then we will be close to Gauss-Bonnet: the line integral of X along the boundary will then be related with an integral of geodesic curvature so that we will reach the local Gauss-Bonnet theorem $\int_M X = \int_M K\, dV = \int_C dt - \kappa_g ds = 2\pi - \int_C dX$. And then by gluing, we will get the **global Gauss-Bonnet theorem** $\int_M X = 2\pi\chi(M)$. This is the mountain peak we wanted to reach. We are in the middle of the climb right now.

DIFFERENTIAL GEOMETRY

MATH 136

Unit 14: Theorema Egregium

14.1. In 1827, Karl Friedrich Gauss proved the **"theorema egregium"**. Is curvature determined by distance measurements within the geometry alone, without reference to the ambient space $\mathbb{R}^3$ in which M is embedded? The answer is yes:

Theorem 1 (Theorema Egegium). *Gaussian curvature K is determined by I.*

FIGURE 1. 1794 Map by mathematician Sam Dunn, when Gauss was 17.

14.2. This implies that if two spaces have different curvature, they can not be isometric. That curvature in a surface M can be expressed using the first fundamental form I is not surprising, given that the first fundamental form is used for all distance measurements in M using geodesics curves like light beams. But curvature has been defined as $\det(A)$, using the **shape operator** $A = I^{-1}II$. It invoked both the first and second fundamental form II that made use of the normal vector n in $\mathbb{R}^3$.

14.3. Last time, we saw that we can assign an **orthonormal frame field** $\{z, w\}$ on M which only depends on the first fundamental form. This produces an orthonormal frame $\{z, w, n\}$ at every point $p \in M \subset \mathbb{R}^3$. This "frame field" on M is similar to the Frenet frame field $\{T, N, B\} = \{e_1, e_2, e_3\}$ on a curve, where the Frenet equations told how the frame field moves with time. We are interested in how the field changes when we change u and v. The mathematics is very similar to what we did for curves just that we have now two variables u, v rather than only one variable t. We will need the following formulas to relate the curl of $X = [P, Q] = [z \cdot w_u, z \cdot w_v]$ with curvature.

Lemma 1 (Moving frame lemma).

$$\begin{aligned} z_u &= (z_u \cdot w)w + (z_u \cdot n)n \\ z_v &= (z_v \cdot w)w + (z_v \cdot n)n \\ w_u &= (w_u \cdot z)z + (w_u \cdot n)n \\ w_v &= (w_v \cdot z)z + (w_v \cdot n)n \end{aligned}$$

Proof. We expand each of the vectors z_u, z_v, w_u, w_v in the $\{z, w, n\}$ basis:

$$\begin{aligned} z_u &= (z_u \cdot z)z + (z_u \cdot w)w + (z_u \cdot n)n \\ z_v &= (z_u \cdot z)z + (z_v \cdot w)w + (z_v \cdot n)n \\ w_u &= (w_u \cdot z)z + (w_u \cdot w)w + (w_u \cdot n)n \\ w_v &= (w_v \cdot z)z + (w_v \cdot w)w + (w_v \cdot n)n \end{aligned}$$

and note that $z \cdot z = 1$ implies $z_u \cdot z = z_v \cdot z = 0$. □

Lemma 2 (X is intrinsic). *$X = [P, Q] = [z \cdot w_u, z \cdot w_v]$ is expressible by I alone.*

Proof. **(i)** Let us look at $z \cdot w_u = -(z_u \cdot w)$: We have seen that $z = ar_u, w = br_u + cr_v$, where a, b, c depended only on I. Now $(z_u \cdot w) = (ar_u)_u \cdot w = (a_u r_u + ar_{uu}, br_u + cr_v)$. Multiply out and use $(r_u \cdot r_u) = E$, and $(r_{uu}, r_u) = E_u/2$ and $(r_{uu} \cdot r_v) = F_u - E_v/2$. All these terms involve entries E, F, G in the first fundamental form I.
(ii) Now do the computation for the second coordinate $z \cdot w_v$. Follow the same steps. You do that in the homework. □

Lemma 3 (Curvature is a Curl). *$dX = Q_u - P_v = (z \cdot w_v)_u - (z \cdot w_u)_v = K\sqrt{\det(g)}$.*

Proof. The wall is climbed in three pitches:
(Pitch i) $\boxed{(z \cdot w_v)_u - (z \cdot w_u)_v = z_u \cdot w_v - z_v \cdot w_u.}$
Proof. Use the product rule and Clairaut's result $z_{uv} = z_{vu}$.
(Pitch ii) $\boxed{z_u \cdot w_v - z_v \cdot w_u = (n_u \times n_v) \cdot n}$
Proof: Use the **moving frame lemma**, the "purple lemma" from the Frenet lecture, as well as $(a \times b)(c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$ to get

$$\begin{aligned} z_u \cdot w_v - z_v \cdot w_u &= (n_u \cdot z)(n \cdot w_v) - (w_u \cdot n)(z_v \cdot n) \\ &= (n \cdot z_u)(n_v \cdot w) - (w \cdot n_u)(z \cdot n_v) \\ &= (n_u \times n_v, z \times w) = (n_u \times n_v, n) \ . \end{aligned}$$

More details in class.
(Pitch iii) $\boxed{(n_u \times n_v) \cdot n = K\sqrt{\det(I)}}$
Proof. The left hand side is (remember $drA = -dn$ defined the shape operator), $((A_{11}r_u + A_{21}r_v) \times (A_{12}r_u + A_{22}r_v)) \cdot n = \det(A)|r_u \times r_v| = \det(A)\sqrt{\det(I)} = K\sqrt{\det(I)}$. □

14.4. The "Theorema Egregium" is proven: the 1-form X is intrinsic. So, the curl dX and also $K(x)$ are intrinsic. You don't see much of what we did here in the literature [1] because one can "shoot down" the Theorema Egregium using explicit formula of K in terms of the intrinsic Γ_{ijk}. What was done here will be used next week however too.

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[1] A literature list consulted for writing these notes will be provided at the end.

Unit 15: Local Gauss-Bonnet

15.1. We now prove the Gauss-Bonnet theorem in the situation when $U \subset R$ is a polygon. The parametrization $r : R \to M$ plants the polygon $r(U) \subset r(R)$ into the surface M.

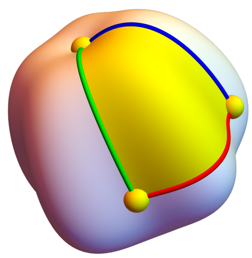

FIGURE 1. The local Gauss-Bonnet theorem tells that face, edge and vertex curvatures of a polygon $r(U)$ in a manifold M add up to 2π.

15.2. A **simple polygon** in M is the image $r(U)$ of a simple polygon $U \subset \mathbb{R}^2$ such that r is smooth and injective on U. Its **Euler characteristic** is $\chi(U) = |V| - |E| + |F| = 3 - 3 + 1 = 1$. As in the discrete Hopf Umlaufsatz, the vertex curvatures are defined as $\kappa_i = \pi - \alpha_i$, where α_i are the polygon angles. The **angles α_i of the polygon** are defined by $\cos(\alpha_i) = \dot{x}_i(1) \cdot \dot{x}_{i+1}(0)$, the dot product of the velocity vectors of the arcs at the end of the incoming and the beginning of the outgoing arc.

15.3. Let U be a simple polygon on M. There are three contributions to curvature: the **face curvature** is the integral of K over the interior, the **geodesic curvature** integrates sectional curvature κ_g over the edges C_j and then there are the **vertex curvatures** $\kappa_j = \pi - \alpha_j$ attached to the vertices.

Theorem 1 (Local Gauss-Bonnet). $\iint_U K dV + \sum_j \int_{C_j} \kappa_g(x_j(t))\, dt + \sum_j \kappa_j = 2\pi$.

15.4. If $x(t) = r(u(t), v(t))$ parametrizes the boundary of the surface $M = r(U)$, we can assume that it is parametrized by arc-length. The velocity vector $\dot{x}$ is a 3-vector tangent to the surface. We look at the orthonormal frame field (z, w) from last time. The **geodesic curvature** of a curve x is defined at points where x is smooth and given as $\boxed{\kappa_g = (n \times \dot{x}) \cdot \ddot{x}}$. Unlike $\kappa = |\dot{x} \times \ddot{x}|$, it is signed. So is the **normal curvature** $\boxed{\kappa_n = n \cdot \ddot{x}}$. Since $\dot{x} \cdot \ddot{x} = 0$, Pythagoras gives $\kappa_g^2 + \kappa_n^2 = \kappa^2$. The velocity vector of the curve can be expressed as an angle so that $\boxed{\dot{x} = \cos(\theta)z + \sin(\theta)w}$. We write $\dot{w}$ for $\frac{d}{dt}w(x(t))$.

Lemma 1 (Geodesic lemma). $\kappa_g = \dot{\theta} - (z \cdot \dot{w})$.

Proof. Fill in the parts of the definition $\kappa_g = (n \times \dot{x}) \cdot \ddot{x}$:
(i) $n \times \dot{x} = \cos(\theta)w - \sin(\theta)z$.
(ii) $\ddot{x} = \dot{\theta}(-\sin(\theta)z + \cos(\theta)w) + \cos(\theta)\dot{z} + \sin(\theta)\dot{w}$.
(iii) So, $\kappa_g = (n \times \dot{x}) \cdot \ddot{x} = \dot{\theta} - z \cdot \dot{w}$ □

15.5. We can now prove the local Gauss-Bonnet theorem:

Proof. (i) Integrating the geodesic lemma gives

$$\int_0^L \kappa_g \, dt = \int_0^L \dot{x} \, dt - \int X \, dr$$

(ii) Green's theorem assures that $\int X \, dr = \iint_U K \, dV$ as $KdV = dX$.
(iii) The Hopf Umlaufsatz for curved polygons gives $\int_0^L \dot{\theta}(t) \, dt + \sum_j (\pi - \alpha_j) = 2\pi$.
(iv) Putting (i),(ii),(iii) together gives the proof. □

15.6. Example 1) If K is constant 0 and U is a triangle, Gauss Bonnet is $\kappa_1+\kappa_2+\kappa_3 = 2\pi$. This is equivalent to $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ you know from elementary school geometry. For an **n-gon**, the identity $\sum_{i=1}^n \kappa_i = 2\pi$ is equivalent to $\sum_{i=1}^n \alpha_i = (n-2)\pi$.

15.7. Example 2) If $M = \mathbb{S}^2$ is a sphere of radius 1, then curvature is $K = 1$. The integral $\iint_U K \, dV$ is the **area** $|U|$ **of the triangle**. The formula becomes $|U| + \sum(\pi - \alpha_i) = 2\pi$ and so $\boxed{\alpha_1 + \alpha_2 + \alpha_3 = |r(U)| + \pi}$. This is **Girard's theorem** or **Harriot's theorem** in spherical geometry, named after Albert Girard or Thomas Harriot.

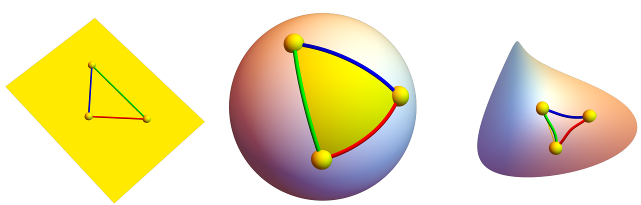

FIGURE 2. A triangle in the plane has $\alpha_1+\alpha_2+\alpha_3 = \pi$. For a spherical triangle of area A, Harriet's theorem gives $\alpha_1 + \alpha_2 + \alpha_3 = \pi + A$. On a hyperbolic space, Lambert's theorem is $\alpha_1 + \alpha_2 + \alpha_3 = \pi - A$.

15.8. Example 3) If M is a surface of constant curvature -1, a triangle is called **hyperbolic** Now, $\iint_U KdV = -|U|$ and $\alpha_1 + \alpha_2 + \alpha_3 = \pi - |U|$, a formula found by Johann Heinrich Lambert. The right hand side $\pi - |U|$ is called **spherical defect**.

15.9. Example 4) Take a sphere with a simple closed geodesic on it, integral of K on each half is 2π. The total integral is 4π.

15.10. Example 5) If $K = 0$ and $r(U)$ is a region in the plane bound by a simple smooth curve, we have the **Hopf Umlaufsatz** $\int \kappa_g(t) \, dt = 2\pi$.

Unit 16: Global Gauss-Bonnet

16.1. Now we are ready to prove the **global Gauss-Bonnet theorem** for a 2-manifold M without boundary. The surface M is triangulated by a **discrete manifold** $G = (V, E, F)$, where the faces F are the triangles defined by the graph (V, E). The discrete manifold is geometrically realized in M as a collection of points, a collection of curves connecting vertices. The geometrically realized network divides M up into triangular faces $M_i = r(U_i)$. The Euler characteristic of M is $\chi(M) = V - E + F$. As we have seen, $\chi(M)$ does not depend on the triangulation: topological changes like removing a disc and gluing in a new disc (we called this as a **connected sum** $M \to M\#S^2$) or doing a **Barycentric refinement** does not change $\chi(M)$.

Theorem 1 (Gauss-Bonnet theorem)**.** *For a compact 2-manifold,* $\iint_M K dV = 2\pi\chi(M)$.

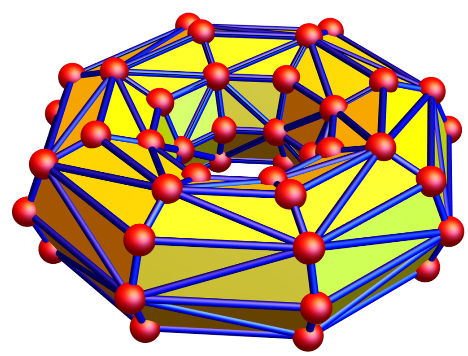

Figure 1. A triangulated manifold M. We apply the local Gauss-Bonnet theorem on each of the triangles. The edge contributions produce with Green's theorem the face curvatures $\iint_{M_i} K \ dV$ as well as $2\pi F$. The vertex contributions produce, using the Euler Handshake lemma, $2\pi(V - E)$. Overall, we have $2\pi(V - E + F) = 2\pi\chi(M)$. The picture shows a $M = \mathbb{T}^2$ with $V = 64$ vertices.

16.2. We will use the local Gauss-Bonnet theorem for each triangle U_i with angles $\alpha_{i1}, \alpha_{i2}, \alpha_{i3}$. We first of all want to understand what happens if we glue together triangles such that the frame field $X = [zw_u, zw_v]$ can be define on the union. On the curve obtained by intersecting two adjacent triangles, the line integral of X cancels.

Lemma 1 (Cancellation). *If two triangles M_1, M_2 meet in a curve C and C_1, C_2 are the parametrizations matching the M_1, M_2, then $\int_{C_1} X\ dr + \int_{C_2} X\ dr = 0$.*

Proof. If X is a 1-form and C is a curve and $-C$ is the curve passed backwards, then $\int_C X\ dr + \int_{-C} X\ dr$. This is what happens here. You can see the identity also as a consequence of Green also, noting that $C \cup -C$ encloses an "empty region". [1] □

16.3. We see that all the 1-form contributions from the edges are zero. The contributions from the faces M_i add up:

Lemma 2 (Additivity). *$\int_M K\ dV = \sum_i \int_{M_i} K\ dV$.*

Proof. The patches M_i are all disjoint. Their union is $\bigcup_i M_i = M$. Areas of disjoint regions add up. □

16.4. The contributions from the vertex degrees $d_i = |S(v_i)|$ add up too. [2]

Lemma 3 (Euler handshake). *If (V, E) has vertex degrees d_i, then $2E = \sum_i d_i$.*

Proof. You prove this in a homework. □

16.5. We still have to look at the contributions from the vertices. At each point P_i we have angles α_{ij} for $j = 1, \dots, d_j$, where d_j is the vertex degree.

Lemma 4 (Adding vertex curvatures). *$\sum_{i=1}^F \sum_{j=1}^{d_i} \kappa_{ij} = 2\pi E - 2\pi V$.*

Proof. Three comfortable pitches (this is called "Genusskletterei").
(Pitch i) $\sum_{i=1}^F \sum_{j=1}^{d_i} \kappa_{ij} = \sum_{i=1}^F \sum_{j=1}^3 (\pi - \alpha_{ij})$.
(Pitch ii) $\sum_{k=1}^V \sum_{j=1}^{d_k} \pi = 2\pi E$.
(Pitch iii) $\sum_{k=1}^V \sum_{j=1}^{d_k} \alpha_{kj} = 2\pi V$. □

16.6. Proof of the global Gauss-Bonnet theorem:
The local Gauss-Bonnet theorem told us $\sum_i [\iint_{U_i} K dV + \sum_j \kappa_{ij} - 2\pi] = 0$. This means that $\iint_U K\ dV + \sum_{i,j} \kappa_{ij} - 2\pi F = 0$. Therefore, using the previous lemma:

$$\iint_U K\ dV = 2\pi V - 2\pi E + 2\pi F = 2\pi\chi(M)\ .$$

16.7. 1) If M is $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, then $\iint_M K\ dV = 4\pi$ (HW).
2) A genus k surface has $\iint_M K\ dV = 2\pi(2 - 2k)$. For a torus $\iint_M K\ dV = 0$.
3) A Klein bottle is obtained by gluing two Möbius strips together. $\iint_M K\ dV = 0$. Because each Möbius strip has Euler characteristic 0 (you computed that in an example), and the Möbius strip can be realized so that the boundary curvature χ_g is zero.

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[1] An easier way is to assume that the connections geodesics so that the line integrals are zero anyway. That works also in the non-orientable case

[2] If V, E, F are the vertices, edges and faces. It is custom to write its cardinalities as V, E, F.

DIFFERENTIAL GEOMETRY

MATH 136

Unit 17: Riemannian Manifolds

17.1. A **locally Euclidean space** M **of dimension** m is a subset of some $\mathbb{R}^n$ such that every $x \in M$ has a neighborhood U, that is homeomorphic to an open subset $R = \phi(U)$ of $\mathbb{R}^m$. The pair (U, ϕ) is called a **chart** producing a **coordinate system** on U: there is a parametrization $r(\phi(x)) = x$, which is a regular map from $R \subset \mathbb{R}^m$ to $\mathbb{R}^n$, meaning that dr has rank m everywhere. A C^k **atlas** on a locally Euclidean space M is a collection $\mathcal{F} = \{U_i, \phi_i\}_{i \in I}$ of charts such that $\bigcup_{i \in I} U_i = M$, and that all $\phi_{ij} = \phi_i \circ \phi_j^{-1}$ are in $C^k(\phi_i(U_j \cap U_i), \mathbb{R}^n)$. An atlas is called **maximal**, if (U, ϕ) is a chart such that $\phi \circ \phi_i^{-1}$ and $\phi_i \circ \phi^{-1}$ are C^k for all $i \in I$, then $(U, \phi) \in \mathcal{F}$. Two atlases $\mathcal{F}, \mathcal{G}$ are called **equivalent** if their union $\mathcal{F} \cup \mathcal{G}$ is an atlas. Given an atlas $\mathcal{F}$, the union of all atlases equivalent to $\mathcal{F}$ is called a **differentiable structure generated by** $\mathcal{A}$. [1]

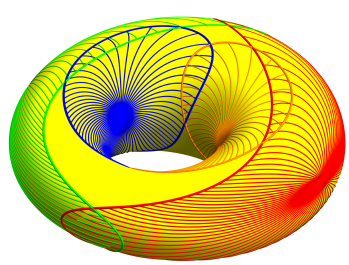

FIGURE 1. A m-manifold $M \subset \mathbb{R}^n$ is shown with part of an atlas $\mathcal{F}$. Each patch U_i is a regularly parametrized by $r : R_i \to U_i$ with $R_i = \phi_i(U_i) \subset \mathbb{R}^m$. The map r has maximal rank m everywhere on R_i.

17.2. A m-dimensional C^k**-differentiable manifold** is a pair $(M, \mathcal{F})$, where M is a m-dimensional locally Euclidean space and where $\mathcal{F}$ is a differentiable C^k structure on M. What this means is that near every point we are in a parametrized region $r(U)$

[1]The concept can be difficult: $\mathcal{F}$ is not unique in general. On S^7, there are 28 different smooth structures. The smooth Poincare conjecture claims that S^4 has a unique differentiable structure.

and so can use what we have done in this course like forming r_u, r_v, define fundamental forms etc. Instead of C^k, we usually just say **smooth**. [2]

17.3. If $E = \mathbb{R}^m$ is the space of **column vectors** of dimension m, its **dual** E^* is defined as the space of all linear maps $f : E \to \mathbb{R}$, It is the space of **row vectors**. If $\{e_1, \dots, e_m\}$ is a basis of E, then $\{e^1, \dots, e^m\}$ denotes a basis of E^*. Every element in E can be written as $v = \sum_i v^i e_i$, every element in E^* can be written as $v = \sum_i v_i e^i$. For $p, q \geq 0$, the linear space T^p_q of all multi-linear maps $(E^*)^p \times E^q \to \mathbb{R}$ is called the space of **tensors of type** (p, q). Column vectors are $(1, 0)$-tensors in $T^1_0 = E$, while row vectors are $(0, 1)$-tensors in $T^0_1 = E^*$, bilinear maps are $(0, 2)$ tensors in T^0_2. A **tensor field of type** (p, q) on a m-manifold M is a smooth assignment of a (p, q) tensor to every point. Such a map is also called a **section** of the tensor bundle, generalizing that a **vector field** is a section of the **tangent bundle** TM. For a $(0, 2)$ tensor field g for example smooth means that for any vector fields X, Y, the function $x \to g(x)(X(x), Y(x))$ is smooth. If $f : M \to \mathbb{R}^k$ is a smooth map, then df is a $(0, 1)$ tensor field. This is also called a **1-form**. A **vector field** means a $(1, 0)$ tensor field. The first fundamental form g is by definition a $(2, 0)$ tensor field, a bilinear form attached to every point. A **Riemannian manifold** (M, g) is a smooth manifold M with a positive definite symmetric $(2, 0)$ tensor field g.

17.4. Let M be a m-manifold and $f : M \to \mathbb{R}^k$ be smooth. A point $x \in M$ is called a **critical point** and $f(x)$ a **critical value**, if the rank of $df(x)$ is not m. Non-critical points are called **regular points**.

Theorem 1. *If M is a m-manifold and $f : M \to \mathbb{R}^k$ is smooth and y is a regular value, then $M_f = f^{-1}(y)$ is a manifold of dimension $m - k$.*

Proof. If $x \in f^{-1}(y)$ is given, the Jacobean map $df(x)$ has rank k and the kernel $H = \ker(df)$ of $df(x)$ is $(m - k)$-dimensional and $H^\perp$ is k dimensional. Take a chart (U, ϕ) in M which contains x. Define

$$g = f \circ \phi^{-1} : \phi(U) \subset \mathbb{R}^m \to \mathbb{R}^k .$$

The projection $L : \mathbb{R}^m = H \oplus H^\perp \to \mathbb{R}^{m-k}, (h, h') \mapsto h'$ onto the orthogonal complement which is non-singular on H. The map $F : \phi(U) \to \mathbb{R}^n \times \mathbb{R}^{m-k}$ as $F(z) = (g(z), Lz)$ has derivative $dF(u) = (dg(u), Lu)$ and which is nonsingular. By the **inverse function theorem**, a neighborhood $\phi(U)$ of $\phi(x)$ is mapped by the diffeomorphism F onto a neighborhood $F(\phi(U))$ of $F(\phi(x))$. We get so a chart $U_x = \phi^{-1} \circ F^{-1}(F(\phi(U)) \cap \{\psi(y)\} \times \mathbb{R}^{m-k})$ on $f^{-1}(y)$ which is mapped by $\phi_x = F \circ \phi$ into a $(m - k)$-dimensional space. Doing the same construction at any point $x \in M$ produces an atlas for $f^{-1}(y)$ and verifies that $f^{-1}(y)$ is a manifold. $\square$

17.5. Examples: a) The $\mathbf{d-sphere}$ is the set $M = S^d = \{x \in \mathbb{R}^{d+1} \mid x_1^2 + x_2^2 + \cdots + x_{d+1}^2 = 1\} \subset \mathbb{R}^{d+1}$. Take two points $A = (0, \dots, 0, 1)$ and $B = (0, \dots, 0, -1)$. The **standard differentiable structure** $\mathcal{F}$ on S^d is generated by $\mathcal{F} = \{(S^d \setminus \{A\}, \phi_A), (S^d \setminus \{B\}, \phi_B)\}$, where ϕ_A are the **stereographic projections** from A to $\{x_{d+1} = 0\}$. b) The set $SL(n, \mathbb{R})$ of $n \times n$ matrices of determinant 1 is a manifold.

[2] A theorem of Whitney assures that any smooth compact m-manifold M (defined more abstractly using fancy-schmancy paracompact Hausdorff spaces) is part of $\mathbb{R}^n$ with $n = 2m + 1$.

DIFFERENTIAL GEOMETRY

MATH 136

Unit 18: Discrete Manifolds

18.1. A **discrete m-manifold** is a finite graph $G = (V, E)$ for which every unit sphere $S(v)$ is a discrete $(m-1)$-sphere. A **discrete m-sphere** is a discrete m-manifold which has the property that removing a point renders it contractible. Inductively, a graph is called **contractible**, if both $S(v)$ and $S \setminus v$ are contractible for some $v \in V$. The 1-point space 1 is contractible. The empty graph is the (-1)-sphere. Let F_k denote the set of K_{k+1} subgraphs (k-simplices) and $f_k = |F_k|$. We have $F_0 = V, F_1 = E$. The **Euler characteristic** of M is defined as $\chi(M) = \sum_{k=0}^{m} (-1)^k f_k = f_0 - f_1 + f_2 - f_3 + \cdots + (-1)^m f_m$. This definition of Ludwig Schläfli generalizes $\chi(M) = f_0 - f_1 + f_2 = V - E + F$ for 2-manifolds.

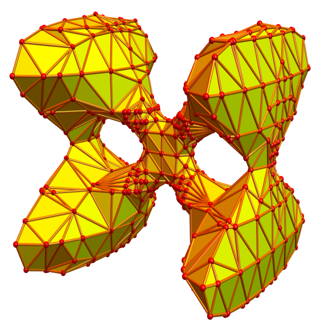

FIGURE 1. This 2-manifold M of genus $g = 2$ has $\chi(M) = 2 - 2g = -2$.

18.2. A graph without edges is a 0-manifold. A 0-manifold is a 0-sphere, if $V = 2, E = 0$ (removing a vertex produces K_1 which is contractible by definition). Every connected 1-manifold is a 1-sphere, a circular graph C_n with $n \geq 4$. Every finite 2-manifold is either a 2-sphere S^2 or a connected sum of tori or projective planes: $M = S^2, M = \mathbb{T}^2 \# \cdots \# \mathbb{T}^2$ or $M = \mathbb{P}^2 \# \cdots \# \mathbb{P}^2$. A 2-sphere can be characterized as 2-manifold of Euler characteristic 2. The 16 cell and the 600 cells are examples of 3-spheres. The join of two 1-spheres is a 3-sphere. The join of a k-sphere with a m-sphere is a (k+m+1)-sphere. The join of G with the 0-sphere is called **suspension**.

18.3. Euler's formula $\chi(M) = V - E + F = 2$ for 2-spheres generalizes to higher dimension. The 0-sphere has $\chi(M) = V = 2$, every 1-sphere has $\chi(M) = V - E = 0$. Every 2-sphere has $\chi(M) = V - E + F = 2$. This pattern continues:

Theorem 1 (Euler's Gem)**.** *If M is a m-sphere, then $\chi(M) = 1 + (-1)^m$.*

Proof. Use induction with respect to dimension m. For $m = 0$, we have $\chi(M) = 2$. The induction assumption is that all $(m-1)$-spheres S satisfy $\chi(S) = 1 + (-1)^{m-1}$. Pick a vertex v. As the unit sphere $S(v)$ is a $(m-1)$-sphere and $S(v) = B(v) \cap G \setminus v$, where both the unit ball $B(v)$ and $G \setminus v$ are contractible with Euler characteristic 1, we have, using the induction assumption, $\chi(M) = \chi(G \setminus v) + \chi(B(v)) - \chi(G \setminus v \cap B(v)) = 2 - (1 - (-1)^{m-1}) = 1 + (-1)^m$. ☐

18.4. In the continuum, manifolds can be constructed as level surfaces of functions like $x^2 + y^2 + z^2 = 1$. We can do that also in the discrete. Take an arbitrary function on vertices V which takes values in $Z_k = \{0, \dots, k\}$. It defines a new graph M_f, where the vertices are the set of complete subgraphs on which f attains all k values. Connect two of these points by an edge, if one is contained in the other. The new graph M_f is a sub-graph of the **Barycentric refinement** of M. Here is the analog of what we have seen classically for functions on manifolds. It surprises that singularities like in the Viviani curve (HW 1) do never occur in the discrete:

Theorem 2 (Level Sets)**.** *If M is a m-manifold and $f : M \to Z_k$ is an arbitrary function, then either M_f is empty or then M_f is a $(m-k)$-manifold.*

Proof. Let x be a n-simplex on which f takes all values. This means $f(x) = Z_k$. The graph $S^-(x) = \{y \subset x, y \neq x\}$ is a $(n-1)$-sphere in the Barycentric refinement of M. The simplices in $S^-(x)$ on which f still reaches Z_k is by induction a $(n-1-k)$-manifold and since we are in a simplex, it has to be a $(n-1-k)$-sphere. Every unit sphere $S(x)$ in the Barycentric refinement is a $(m-1)$-sphere as it is the join of $S^-(x)$ with $S^+(x) = \{y, x \subset y, x \neq y\}$. (The join of two spheres is always a sphere.) The sphere $S^+_f(x)$ in M_f is the same than $S^+(x)$ in M because every simplex z in M containing x automatically has the property that $f(z) = Z_k$. So, the unit sphere $S(x)$ in M_f is the join of a $(n-k-1)$-sphere and the $(m-n-1)$-sphere and so a $(m-k-1)$-sphere. Having shown that every unit sphere in M_f is a $(m-k-1)$-sphere, we see that M_f is a $(m-k)$-manifold. ☐

18.5. What about differential geometry? No problem. Define **curvature** as

$$K(v) = \sum_{k=0}^{m} \frac{(-1)^k f_{k-1}(S(v))}{k+1} = 1 - \frac{f_0(S(v))}{2} - \frac{f_1(S(v)))}{3} + \dots .$$

In the case of a 2-manifold this boils down to $1 - f_0(S(v))/2 + f_1(S(v))/3 = 1 - d(v)/6$, where $d(v)$ is the vertex degree. For odd-dimensional manifolds, the curvature is constant zero. You experiment with this in Homework 10.

Theorem 3 (General Gauss-Bonnet)**.** $\sum_{v \in V} K(v) = \chi(M)$

Proof. The proof is the same as in the 2-dimensional case. Again look at the energies $\omega(x) = (-1)^{\dim(x)}$ attached to each simplex x in the graph (complete subgraph with $\dim(x) + 1$ vertices). Then $\chi(M) = \sum_x \omega(x)$. Now distribute all these energies of a k-simplex x equally to the $k+1$ vertices contained in x. As there are f_{k-1} simplices in $S(v)$ which correspond do simplices containing v, this adds $\frac{(-1)^k f_{k-1}(S(v))}{k+1}$ to each vertex v. Now just collect all up at a vertex v to get the curvature $K(v)$. The transactions of energies preserved the total energy = Euler characteristic. ☐

Unit 19: Curvature Tensor

19.1. While $\Gamma_{ijk} = \frac{1}{2}[\frac{\partial}{\partial u^i} g_{jk} + \frac{\partial}{\partial u^j} g_{ki} - \frac{\partial}{\partial u^k} g_{ij}]$ is not a tensor, the **Riemann curvature**

$$R^s_{ikj} = \frac{\partial}{\partial u^k}\Gamma^s_{ij} - \frac{\partial}{\partial u^j}\Gamma^s_{ik} + \sum_r \Gamma^r_{ij}\Gamma^s_{rk} - \sum_r \Gamma^r_{ik}\Gamma^s_{rj}$$

is a $(1,3)$ tensor. Think of it as a matrix R^s_i describing a linear transformation when rotating around a small square in the k,j plane. We can also look at at the $(0,4)$-tensor $R_{mikj} = \sum_s g_{ms} R^s_{ijk}$.

19.2. To gain more intuition, let us rewrite this without coordinates. Let $X = \sum_i X^i e_i$ and $Y = \sum_i Y^i e_i$ be vector fields (1,0) tensor fields. [1] Reflected in the notation $e_i = \partial_{u^i}$ is that a vector field $X = \sum_i X^i e_i$ also defines a linear map on functions $Xf = \sum_i x^i f_{u^i} = df X$, the **directional derivative**. [2] Since every vector field X also is a linear map, one can look at the commutator $[X,Y] = XY - YX$, which is by Leibniz again a vector field. Proof: in coordinates $X = \sum_j X^j e_j, Y^j = \sum_j Y^j e_j$, this **Lie bracket** is $[X,Y]^i = \sum_j X^j \partial_j Y^i - Y^j \partial_j X^i$.

19.3. The **covariant derivative** $\nabla_X Y$ is a new vector field. Axiomatically it is determined by **Leibniz** $\nabla_X(fY) = (Xf)Y + f\nabla_X Y$, **metric compatibility** $g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = Xg(Y,Z)$, and being **torsion free** $\nabla_X Y - \nabla_Y X = [X,Y]$. The **fundamental theorem of Riemannian geometry** assures that there exactly one such derivative: and this is $\nabla_{e_i} e_j = \sum_k \Gamma^k_{ij} e_k$ determining the Riemann curvature tensor R.

19.4. The curvature tensor now also can be written as $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ meaning $g(e_s, R(e_k, e_j)e_i) = \sum_r g_{sr} R^r_{ikj} e_r = R_{sikj}$. Intuitively, $R(e_k, e_j)$ tells what happens if one parallel transports along a small rectangular loop spanned by e_k, e_j. A linear transformation A^s_i results from looping in the k,j plane.

19.5. For linearly independent vectors u, v, the **sectional curvature**

$$K(u,v) = \frac{g(R(u,v)v,u)}{g(u,u)g(v,v) - g(u,v)^2}$$

is probably the most intuitive approach to curvature as it does not depend on the coordinate system. It only depends on the plane defined by the tangent vectors u, v. If M is two dimensional, it agrees with the Gauss curvature. But this is not obvious as it reestablishes the Theorema egregium!

[1] Most of the literature uses capital letters for vector fields. $\sum_i X^i e_i$ rather than $\sum_i v^i e_i$.

[2] The directional derivative is written as $\nabla f \cdot v$ in multi-variable calculus.

Theorem 1. *The sectional curvatures determine the Riemann curvature tensor.*

19.6. The **Ricci curvature** R is a contraction of the curvature tensor $\boxed{R_{ik} = \sum_j R^j_{ijk}}$.

The **scalar curvature** is then the contraction of the Ricci curvature $\boxed{S = \sum_{j,k} g^{jk} R_{jk}}$. In two dimensions, it is twice the Gauss curvature. The **Einstein tensor** G is defined as $\boxed{R - Sg/2}$. A metric is called an **Einstein metric** if $R = \lambda g$ for some λ. Define the **Hilbert functional** $S(g) = \int_M S_g dV_g$ and the inner product on $(0,2)$ tensors as $\langle a, b\rangle_g = \int_M \sum_{i,j} a(e_i, e_j) b(e_i, e_j)\ dV$. Under which conditions is Hilbert functional extremal? [3]

Theorem 2. $\frac{d}{dt} S(g + th) = \langle Sg/2 - R), h\rangle_g$.

Theorem 3. *Every 2-manifold is an Einstein manifold: $Sg/2 - R = 0$.*

Proof. The reason is that $K = S/2$ and that the Hilbert functional $S(g) = 2\int_M K dV = 4\pi\chi(M)$ does not depend on the metric by the global Gauss-Bonnet theorem. □

We see that in the 2 dimensional case the Ricci tensor R is K times the Riemannian metric tensor g. Again, this is not obvious as it re-establishes the Theorema egregium.

19.7. In order to prepare for relativity, we also need to generalize Riemannian manifolds. A **metric tensor** on a linear space E is a symmetric $(0,2)$ tensor which is **non-degenerate** that is $g(u,v) = 0, \forall v \in E \Rightarrow u = 0$. A **metric tensor field** g is a tensor field $g \in T^0_2(M)$ such that $g(x)$ is a metric tensor in $T^0_2(T_x M)$. This means that for any vector fields X, Y the function $x \to g(x)(X(x), Y(x))$ is smooth. A **pseudo Riemannian manifold** is a smooth manifold with a metric tensor field g on M. A pseudo Riemannian manifold (M, g) is a **Riemannian manifold**, if g is positive definite, meaning $g(x)(v,v) \geq 0$ for all v. The **length** of a vector $v \in T_p M$ is defined as $||v|| = \sqrt{|g(p)(u,u)|}$, where $g(p)(u,v) = \sum_{ij} g_{ij}(p) u^i v^j$. [4] A vector of length zero is called **null**. Vectors u for which $\sum_{ij} g_{ij} u^i u^j < 0$ are **time like**, vectors u with $\sum_{ij} g_{ij} u^i u^j > 0$ **space like**. The **length** of the curve is defined by $\int_a^b ||\dot{x}(t)||\ dt$.

19.8. Does every manifold allow a pseudo Riemannian manifold of a certain signature?

Theorem 4. *On any Riemannian manifold there exists a Riemannian metric g.*

Proof. There is a tensor field $g \in T^0_2(M)$ which is symmetric, non-degenerate and positive definite: let $\{U_i, \phi_i\}$ be an atlas for M and let p_i be a **partition of unity**, subordinate to the cover U_i. Let q be a Riemannian metric on $\mathbb{R}^n$. For example $[q] = \text{Diag}(1,1,1\dots,1)$. Let $q_i = \phi_i^* q$ be the pull back metrics on U_i. Define $g(p) = \sum_i g_i(p) q_i(p)$. This is smooth and positive definite because for $p \in M$ and u in the tangent space $T_p M$, we have $g(p)(u,u) = \sum_i g_i q_i(u,u) > 0$. □

19.9. It is not always possible to build on a given manifold a metric of a given signature. For example, on the sphere $M = S^2$, there exists **no Lorentzian metric**, that is a metric of signature $(-1, 1)$. The reason is that one can not comb a 2-sphere.

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[3] A proof can be found on pages 312-320 in Kuehnel.

[4] Note the appearance of an absolute value.

Unit 20: Relativity

20.1. The **principle of general covariance** states that the pseudo Riemannian manifold (M, g) alone defines gravitational laws. No preferred basis nor background "aether" concepts are allowed. Physical laws are invariant under smooth coordinate changes. Objects of interests are tensorial. Einsteins theory of gravity links **space-time** with **matter**. Matter determines space time (M, g) in that g by minimizing the Hilbert action. Space time determines the paths of particles by minimizing kinetic action. [1]

The Einstein Equations	The Geodesic Equations
$\boxed{R - \frac{1}{2}Sg + \Lambda g = \kappa T}$	$\boxed{\ddot{x} = -\sum_{i,j} \Gamma_{ij}^k \dot{x}^i \dot{x}^j}$
Matter tells Space-time how to curve	Space-time tells Matter how to move

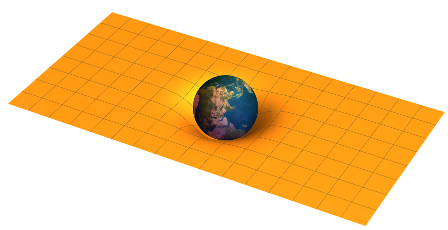

FIGURE 1. Mass deforms space time

20.2. The theory has been experimentally confirmed in various instances like 1) The **perihelion advance of planets like Mercury**, 2) the **gravitational lensing of light** around stars or galaxies, 3) the **time delay in radar probing of planets**, 4) the **spectral shift of light emanating from massive objects**, 5) the **precession of a gyroscope**,freely orbiting the earth, 6) the **detection of gravitational waves** from Black-hole mergers, 7) the **pictures of black holes** like Sagittarius A* and Messier 87* by the **event horizon telescope**.

[1]Even when restricting to gravity and not taking into account quantum mechanics, this is unsatisfactory. In a 2-body problem of two massive particles like a black-hole binary the masses should contribute to the stress-energy tensor T. In black-hole situations, the paths need to be removed from the manifold. About the math: Yvonne Choquet-Bruhat (1923-) dealt with the Cauchy problem in 1969. Demetrios Christodoulou and Sergiu Klainermann proved in 1994 nonlinear gravitational stability. Numerical schemes for black hole binaries in vaccum using **post-Newtonian expansions** exist since 2005. It is a total mess, both from an applied math (engineering) as pure math perspective.

20.3. In **Special relativity** the metric is no more required to be positive definite. The **Galilei group** generated by rotations and translations is replaced by the Poincaré group generated by **Lorentz transformations** and translations. Any rotation matrix in a (e_0, e_i)-plane with $i = 1, 2, 3$ is replaced by a hyperbolic rotation, where cos is replaced by cosh and sin by sinh leading to **Lorentz boosts**. For $M = \mathbb{R}^4$

$$g = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with speed of light c is called the **flat Lorentz metric**. [2] The null vectors are the vectors in the **light cone** $\{||u|| = 0\}$. The vector $(1, 0, 0, 0)$ is time like and $(0, 1, 0, 0)$ is an example of a space like vector. Particles with space like velocity vectors have velocity smaller than the speed of light, particles with time like velocity vectors are called **tachions**. They have velocity larger than the speed of light.

20.4. The most important example in general relativity is the **Schwarzschild metric**. All the known confirmations of general relativity are just based on this model. On the manifold $M = \mathbb{R}^4 \setminus \{r \leq 2m\}$ we can use the spherical coordinates $t = x^0, r = x^1, \theta = x^2, \phi = x^3$. For $r > 2m$, the **Schwarzschild metric** is given by

$$g = \begin{bmatrix} -c^2(1 - \frac{2m}{r}) & 0 & 0 & 0 \\ 0 & (1 - \frac{2m}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2(\phi) \end{bmatrix}.$$

It is a model for the space-time in presence of a single massive object of mass M. [3] The constant m is GM/c^2, where G is the gravitational constant and M the mass. The number $2m$ is called the **Schwarzschild radius**. You verify that (M, g) satisfies the Einstein equations in the homework.

20.5. The following establishes why the theory "makes sense". It is a refinement of the Newton theory of gravitation.

Theorem 1. *Relativity for slow particles in a weak field becomes Newtonian mechanics.*

Proof. If $g = \overline{g} + h$, where $\overline{g}$ is the flat metric and h is small then $\dot{x}^i$ for $i = 1, 2, 3$ can be neglected with respect to $\dot{x}^0$. The geodesic equations $\ddot{x}^k = -\sum_{i,j} \Gamma^k_{ij} \dot{x}^i \dot{x}^j$ is approximated by $-\Gamma^k_{00} = \partial_{u^k} \overline{g}_{00} - \partial_{u^0} h_{0k}$. If the gravitational field does not change in time, the later term goes away and $\ddot{x} = -\frac{1}{2} \nabla h_{00}$ which is the Newtonian equation for $h_{00} = 2V$. We therefore have $g_{00} = 1 + 2V/c^2$. Now V/c^2 is 10^{-9} for the earth, 10^{-6} on the sun, 10^{-1} for a neutron star. □

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[2] Also Misner,Thorne, and Wheeler use g to be positive definite on space-like hyper-surfaces.

[3] By a **theorem of Birkhoff**, the unique spherically symmetric solution of Einstein's equations.

DIFFERENTIAL GEOMETRY

MATH 136

Notations

Algebra

99.1. $\mathbb{R}^n$ is the **Euclidean space** in n dimensions. A finite rectangular array A of real numbers is called a **matrix**. If there are n rows and m columns in A, it is called a $n \times m$ matrix. The set of all theses matrices $\mathbb{R}^{n,m}$ forms a vector space. The entry of A_i^j is in the i'th row and j'th column. A $n \times 1$ matrix is a **column vector = (1,0) tensor**, a $1 \times n$ matrix is a **row vector= (0,1) tensor**. A 1×1 matrix is a **scalar = (0,0) tensor**. For a $n \times p$ matrix A and a $p \times m$ matrix B, the $n \times m$ **matrix product** AB is defined as $(AB)_i^j = \sum_{k=1}^p A_i^k B_k^j$. The **transpose** of a $n \times m$ matrix A is the $m \times n$ matrix $(A^T)_i^j = A_j^i$. The transpose of a column vector is a row vector. There are various types of matrices. Square matrices are the case $n = m$ ((1,1)-tensor), symmetric matrices satisfy $A_i^j = A_j^i$ for all i, j. There is an **inner product** $A \cdot B = \text{tr}(A^T B) = \sum_{i,j} A_i^j B_j^k$ on the space of matrices. With an inner product, we have a length $|A| = \sqrt{A \cdot A}$. The Cauchy-Schwarz inequality is $|A \cdot B| \leq |A||B|$ and allows to define angles $\cos(\alpha) = A \cdot B/(|A||B|)$ between two nonzero matrices of the same type. This generalizes the usual **dot product** for vectors.

Topology

99.2. A **topological space** is a set X on which one has a set of subsets $\mathcal{O}$ called **open sets** in X. One assumes that $\emptyset, X \in \mathcal{O}$, that arbitrary unions from $\mathcal{O}$ are in $\mathcal{O}$ and that finite unions of elements in $\mathcal{O}$ are in $\mathcal{O}$. The complement of an open set is called closed. Sets that are both open and closed are **clopen**. The empty set and X are always clopen. If $x \in U$, then U is also called a **neighborhood** of x. The topological space is called **Hausdorff**, if $x \neq y$ are points in M, then there exist $U, V \in \mathcal{O}$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. A **metric space** (X, d) defines a topological space. Start with the open balls $B_r(x) = \{y, d(x, y) < r\}$ and assume them to be in $\mathcal{O}$. Then take the smallest topological space which contains all these open sets. This is the topology generated by the metric. A metric space is always Hausdorff because $x \neq y$ with $r = d(x, y) > 0$ implies that $B_{r/3}(x) \cap B_{r/3}(y) = \emptyset$.

99.3. Euclidean spaces like $\mathbb{R}^n$ or $\mathbb{R}^{n,m}$ have a natural distance $d(x, y)$ on them, given as the length of the vector from x to y, if the length or magnitude of a vector is defined as $\sqrt{v \cdot v}$. For a matrix this is defined by $d(A, B)^2 = \sum_{i,j}(A_{ij} - B_{ij}^2)$. There are other distance functions like if the dot product is allowed to change from point to point. Euclidean space equipped with a distance becomes a metric space and so have a **topology**, meaning that we can form open sets, which are also called **neighborhoods**. In a metric space the open balls $B_r(P) = \{Q, d(P,Q) < r\}$ are open sets and form a basis for all open sets. Given a topology, we also have a notion of continuity meaning that the inverse of an open set is open. For metric spaces, this is equivalent to $\lim_{x_n \to x} f(x_n) = f(x)$ if a sequence of points x_n converges to x. A subset G of Euclidean space is called open if every point x has an open ball $B_r(x) = \{y, d(x,y) \leq r > 0\}$ which is contained in G. The complement of an open set is called **closed**. A subset of an Euclidean space is called **connected** if it can not be written as a disjoint union of two open sets. In other words, if $\emptyset$ and X are the only clopen sets, then X is connected.

Calculus

99.4. If f is a differentiable map from $\mathbb{R}^m$ to $\mathbb{R}^n$, its derivative df at x is the matrix $df(x) \in R^{n,m}$, the **Jacobean matrix** at x. Its entry is $df_{ij} = \partial f_i / \partial x_j$. For example, if $r : \mathbb{R} \to \mathbb{R}^n$ defines a curve, then r' is a column vector giving the velocity of r. In the case $n = 1$, this is the derivative in calculus. If $r : \mathbb{R}^> \to \mathbb{R}$ is a function of p variables, then $df = [\frac{\partial}{\partial x_1} f, \dots, \frac{\partial}{\partial x_p}]$ is the **gradient**. If $f : \mathbb{R}^p \to \mathbb{R}^n$ is another function with $df(y) \in \mathbb{R}n, p$, we can combine them and form $f \circ r(x) = f(r(x)) : \mathbb{R}^m \to \mathbb{R}^n$. The matrices $df(y) \in \mathbb{R}^{n,p}$ and $dr(x) \in \mathbb{R}^{p,m}$ combine to the matrix product $df\ dr$ at a point, a matrix in $\mathbb{R}^{n,m}$. The **multi-variable chain rule** is: $d(f \circ r)(x) = df(r(x))dr(x)$. As df can again be seen as a map between Euclidean spaces, we can differentiate again if this is possible. If we can differentiate k times and get a continuous function, we say $f \in C^k(\mathbb{R}^m, \mathbb{R}^n)$. In differential geometry, we usually assume that functions are **smooth**, meaning that we can differentiate as many times as we want. If there is a differentiable map $f : M \to N$ for which there is a differentiable inverse map g, then f is called a **diffeomorphism**.

Analysis

99.5. The **implicit function theorem** tells that if $f(x,y)$ is a function of two variables and $\partial_y f$ is invertible at x_0, then $y = g(x)$ near x_0. The **inverse function theorem** tells that if $f : \mathbb{R}^n \to \mathbb{R}^n$ has an invertible Jacobean matrix at x_0, then f is invertible near x_0. It follows that if $f : \mathbb{R}^m \to \mathbb{R}$ is a map such that df is nowhere zero on all points $\{f = 0\}$, then $\{f = 0\}$ is a $(m-1)$-dimensional submanifold. More generally, if $f : \mathbb{R}^n \to \mathbb{R}^k$ has the property that df has maximal rank k at every point of $\{f = c\}$ then $\{f = c\}$ is a $(n-k)$-dimensional sub-manifold. The space $\mathbb{R}^n \times \mathbb{R}^n$ is called the **tangent bundle** of $\mathbb{R}^n$. If M is a k-dimensional submanifold given by $\{f = c\}$ then the tangent space $T_p M$ at a point consists of all points perpendicular to the normal vector df^T. It is the kernel of $\ker(df)$. The normal space $\perp_p M$ is spanned by the vector df^T. For example, if $f(x,y,z) = x^2 + y^2 + z^2$, where $M = \{f = 3\}$ is a sphere and $df = [2x, 2y, 2z]$ is the gradient, the transpose $df^T = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$ is normal the manifold. The tangent space at a point like $P = (1,1,1)$ is the set of vectors perpendicular to the normal vector $[2,2,2]^T$.

Oliver Knill, knill@math.harvard.edu, Math 136, Fall, 2024

DIFFERENTIAL GEOMETRY

MATH 136

Unit 1: Homework

This is the first homework. It is due Friday September 13.

Viviani's curve

Problem 1.1: The curve

$$r(t) = \begin{bmatrix} \cos^2(t) - \frac{1}{2} \\ \sin(t)\cos(t) \\ \sin(t) \end{bmatrix}$$

is called **Viviani's curve**. Verify that this curve is on the intersection of the sphere $f_1(x, y, z) = (x + 1/2)^2 + y^2 + z^2 = 1$ and the cylinder $f_2(x, y) = x^2 + y^2 = 1/4$.

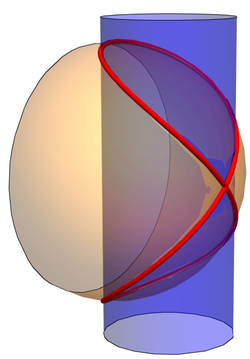

Figure 1. The Viviani curve is the intersection of a sphere and a cylinder.

Problem 1.2: Compute the velocity $r'(t)$ and acceleration $r''(t)$ as well as the jerk $r'''(t)$.
a) Compute the **curvature** of the curve at $t = 0$.
b) Compute the **torsion** of the curve at $t = 0$.

Problem 1.3: Lets look at the map from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by

$$f(x,y,z) = \begin{bmatrix} f_1(x,y,z) \\ f_2(x,y,z) \end{bmatrix} .$$

Viviani's curve can be written as the set $f(x,y,z) = c$ with $c = \begin{bmatrix} 1 \\ 1/4 \end{bmatrix}$.
Compute the **Jacobean matrix**

$$df(x,y,z) = \begin{bmatrix} \frac{\partial}{\partial x} f_1 & \frac{\partial}{\partial y} f_1 & \frac{\partial}{\partial z} f_1 \\ \frac{\partial}{\partial x} f_2 & \frac{\partial}{\partial y} f_2 & \frac{\partial}{\partial z} f_2 \end{bmatrix} .$$

Problem 1.4: Viviani's curve is the intersection of two polynomial equations. It is known as an **algebraic curve**. It is not a manifold however. With a crossing, it is topologically a **figure 8**. By the implicit function theorem, the curve is **manifold like** near any point, where the Jacobean matrix has rank 2 (which in our case means that the gradients are not parallel). Verify that the matrix df has rank 1 at $(1/2, 0, 0)$ and rank 2 everywhere else.

Problem 1.5: a) Lets compute the curvature of a sphere of radius 1 using the Puiseux's formula. The sphere is parametrized as

$$r(u,v) = [\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)] .$$

Let p be a point on the sphere (take the north pole) and $S_r(p)$ the sphere of radius r centered at p. First verify that in the north pole case $S_r(p)$ is parametrized as $r(t) = [\cos(t)\sin(r), \sin(t)\sin(r), \cos(r)]$. Now check that $K(p) = \lim_{r \to 0} 3 \frac{2\pi r - |S_r(p)|)}{\pi r^3} = 1$.
b) Use the same curvature notion to compute it for the cylinder of radius 1/4 appearing in Vivianis's curve.

Remark: Much of our course will be devoted to curvature. We will in the third week look at curvature and torsion and see that they determine a curve up to translation. Later in the course we look at curvature of surfaces defined as $K = \det(II)/\det(I)$ using matrices I, II called first and second fundamental form. This will lead to the Gauss-Bonnet formula. The Puiseux formula for curvature is an intuitive notion of curvature which we will be able to show to be equivalent to the "official curvature". But that is far from obvious. The statement that curvature is an intrinsic notion of a surface and does not depend on any embedding of the surface in space is the "Theorema Egregium". It will be one of the goals of this course to understand this.

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DIFFERENTIAL GEOMETRY

MATH 136

Unit 2-3: Homework

This is the second homework. It is due Friday September 20.

Surfaces

Problem 2.1: The fundamental theorem of linear algebra tells that for any $n \times m$ matrix,

$$\ker(A)^\perp = \mathrm{im}(A^T) .$$

a) Prove this formally and also illustrate your proof with an example.
b) Verify the **rank-nullity theorem**: $\dim(\mathrm{im}(A)) + \dim(\ker(A)) = m$ holds for any $n \times m$ matrix. Use the same example to illustrate your proof.
c) If A is a $n \times m$ matrix and $b \in \mathbb{R}^n$. Under which conditions is $\{x, Ax = b\}$ a k-dimensional manifold?

Problem 2.2: a) Verify that

$$r(\theta, \psi, \phi) = \begin{bmatrix} \cos(\theta)\cos(\phi) \\ \sin(\theta)\cos(\phi) \\ \cos(\psi)\sin(\phi) \\ \sin(\psi)\sin(\phi) \end{bmatrix}$$

parametrizes the 3-sphere $S : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ in $\mathbb{R}^4$. Here $\theta \in [0, 2\pi], \psi \in [0, \pi]$ and $\phi \in [-\pi/2, \pi/2]$. (To verify that this is the right choice of parametrization just check that $\theta \to \theta, \psi \to \psi + \pi, \phi \to -\phi$ produces the same point.)
b) Verify that the set $SU(2)$ of complex 2×2 matrices of the form

$$A = \begin{bmatrix} z & -\overline{w} \\ w & \overline{z} \end{bmatrix}$$

which have determinant 1 also represent the 3-sphere. Verify that there is a $1 - 1$ correspondence between $SU(2)$ and S.
c) Conclude that there is a multiplicative structure $*$ on the 3-sphere. We can define $x * y$ are points in S and get a new point. This multiplication is associative and each element has a unique inverse. Which point on the sphere $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ represents the 1-element?

Problem 3.3: a) Verify that the surface parametrized as

$$r(u,v) = \begin{bmatrix} \cos(u)\cos(v) \\ \sin(u)\cos(v) \\ \cos(u)\sin(v) \\ \sin(u)\sin(v) \end{bmatrix}$$

is contained in the 3-sphere. Check that it is a regular surface and so a 2-manifold by checking that dr has rank 2 everywhere.
b) What kind of surface is it? Compute its surface area with the formula given in the notes. Note that u, v both go from 0 to 2π.

Curves

Problem 2.4: a) Verify that for every non-zero integers a, b the curve

$$r(t) = \begin{bmatrix} \cos(at)\cos(bt) \\ \sin(at)\cos(bt) \\ \cos(at)\sin(bt) \\ \sin(at)\sin(bt) \end{bmatrix}$$

t goes from 0 to 2π is contained in the 3-sphere.
b) Find the arc length using the formula you know. The answer depends on a, b.
c) Give an explicit parametrization of a closed curve in the 3-sphere for which the arc length is larger than 1000.

Problem 2.5: a) Given two points A, B in $\mathbb{R}^n$. Prove the **theorem of Archimedes** telling that the straight line gives the shortest connection between A and B. That is, among all smooth curves connecting A with B. We will later call the shortest connection a **geodesic**.
b) Prove that any smooth regular curve $r : [a,b] \to \mathbb{R}^n$ can be parametrized by arc length s: there is a new parametrization $f(s)$ such that the velocity is $|f'(s)| = 1$ at all points. We will need this result next week, when we prove the **fundamental theorem of curves**.

DIFFERENTIAL GEOMETRY

MATH 136

Unit 4-5 Homework

This is the third homework. It is due Friday, September 27.

Problem 1: Almost all books or courses give as problem number 1 in the course the task to prove the formulas

$$\kappa = |r' \times r''|/|r'|^3, \quad \tau = (r' \times r'') \cdot r'''/|r' \times r''|^2 .$$

We actually will prove these formulas in class. You do not have to reprove them here. What we want you do, is the much easier reverse: give a detailed proof that if $r(t)$ is parametrized by arc length, then these formulas agree with the formulas for curvature and torsion you have seen in class, that is the formulas which define curvature and torsion in the case of constant arc length parametrization.

Problem 2: a) Look up and write down a proof that if $F(t,x)$ is a differentiable function from $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, and $x_0 \in \mathbb{R}^n$, then there exists an open interval $(-a,a)$ and a unique path $x(t) : (-a,a) \to \mathbb{R}^N$ such that $x' = F(t,x)$ and $x(0) = x_0$. We want you to write down the proof in the differentiable case which is a bit more special than the usual assumption assuming a Lipschitz property for F.
b) Justify that if $x(t)$ stays bounded meaning that there is constant such that $|x(t)| \leq M$ for all t, then the solution exists for all t. (We call this a global solution.) Now conclude that if $Q' = K(t)Q$ is a differential equation for a matrix $Q(t)$ with skew symmetric $K(t)$, then there is a global solution.

Problem 3: a) Determine from each of the spaces $SO(n), so(n), SU(n), su(n)$ whether they are linear spaces or not.
b) Check that if $x(t)$ is a differentiable curve in $SO(n)$, then $x(t)$ satisfies the differential equation $x'(t) = A(t)x(t)$, where $A(t) \in so(n)$, the space of skew-symmetric matrices.
c) Show that $A(t) = A$ is a constant skew symmetric matrix, then the **matrix exponential** $Q(t) = e^{At}$ is an orthogonal matrix. What is this matrix $Q(t)$ in the case $n = 2$?

Problem 4: a) First verify that the helix $r(t) = [\cos(at), \sin(at), bt]$ has constant curvature and torsion. What are the values?
b) Now prove that if a curve has constant curvature and torsion, it must be a helix.

Problem 5: a) Verify that if $B(t) \in so(n)$ and $Q' = BQ$, then $L(t) = Q(t)L(0)Q^T(t)$ satisfies the so called **Lax pair** differential equation

$$L' = [B, L] = BL - LB \; .$$

Conclude that the eigenvalues of L are preserved. (This fact an important part of the theory of **integrable systems** which can explain phenomena of solitons.)
b) (*) Let $B(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Write down the Lax pair differential equation and solve it for $L(0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(*) This is probably the simplest non-trivial example of a Lax pair. It had been given by Hermann Flaschka (1945-2021), one of the pioneers in integrable systems in 1974. Flaschka had been the chair of the math department in Tucson when I (Oliver) had been teaching there.

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DIFFERENTIAL GEOMETRY

MATH 136

Unit 6-7 Homework

This is the fourth homework. It is due Friday, October 4rd.

Problem 1: Check the Hopf Umlaufsatz in the example of an asteriod

$$r(t) = [\cos^3(t), \sin^3(t)] .$$

First verify that $|r'(t)| = |3\cos(t)\sin(t)|$ and $\kappa(t) = -(2/3)/|\sin(2t)|$ then compute the rotation index $\int_0^{2\pi} \kappa(t)|r'(t)|\, dt/(2\pi)$. While your result will comply with the Hopf Umlaufsatz, there is something strange going on given how you rotate counterclockwise around the region. Figure it out!

Problem 2: a) Compute the rotation index in the case of the simple closed curve

$$r(t) = 17[\cos(t), \sin(t)] - [\cos(17t), \sin(17t)] .$$

You will see that $\int_0^{2\pi} \kappa(t)|r'(t)|\, dt/(2\pi) = 9$. We have complete melt-down of the Umlaufsatz. Comment on what is going on.
b) Now do the computation when 17 is replaced by 2:

$$r(t) = 2[\cos(t), \sin(t)] - [\cos(2t), \sin(2t)] .$$

You will get the rotation index 3/2. **W**hy **T**his **F**ailure?

Problem 3: A **discrete Hopf Umlaufsatz** can be formulated for polygons with n vertices.
a) Assume first we have a simple convex polygon with n vertices. Define the curvature at the vertex v_k to be κ_k which is the outer angle $\pi - \alpha_k$ where α_k is the angle you have defined in third grade for polygons. The discrete Hopf Umlaufsatz tells $\sum_{k=1}^{n} \kappa_k = 2\pi$. Prove this.
b) Now formulate the general (not necessarily convex) case. Define suitable curvatures such that the result works.

Problem 4: Check the four vertex theorem in the example

$$r(t) = 4[\cos(t), \sin(t)] - [\cos(2t), \sin(2t)] .$$

The expressions for $\kappa(t)$ and $\kappa'(t)$ are not that bad. Plot the function $\kappa(t)$ and find the critical points, the roots of κ'.

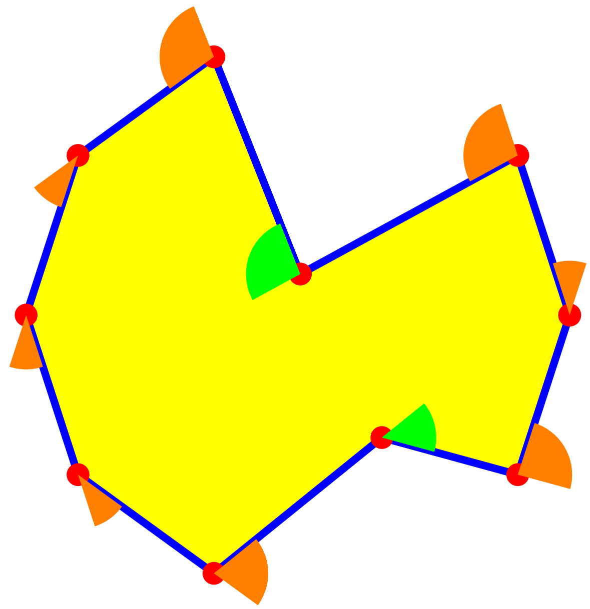

FIGURE 1. Problem 2: Proving the Umlaufsatz for polygons. Curvatures can get positive or negative.

Problem 5: For parameters a, define $c = 2\sqrt{(a)}$ and the curve $r(t) = [a\cos(t)+\cos(3t), a\sin(t)-\sin(3t), c\sin(2t)]$. For $a = 1$ one has the **tennis ball curve**, for $a = 1/2$ the **base ball curve** and for $a = 1.8$, the **basket ball curve** (Basketballs have two additional grand circles).
a) Verify that these curves are located on a sphere.
b) Look up the tennisball theorem, state its content, then write down the main idea on how the theorem is proven.

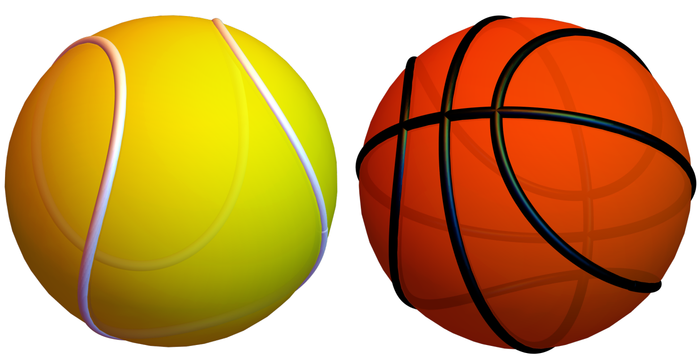

FIGURE 2. Problem 5: A tennis ball and a basket ball.

DIFFERENTIAL GEOMETRY

MATH 136

Unit 8-9 Homework

This is the fifth homework. It is due Friday, October 11rd.

Problem 1: a) Compute the first and second fundamental form for the surface

$$r(u,v) = \begin{bmatrix} u \\ u^2 - v^2 \\ v \end{bmatrix} .$$

In this problem, we do not want you to use a computer algebra system. You need to write down especially all the matrices dr, dn and A.
b) Use a) to compute the curvature and especially the curvature at $(0,0,0)$.

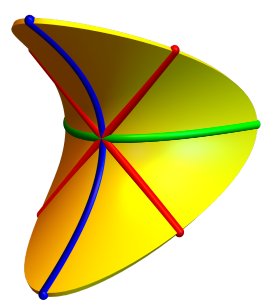

FIGURE 1. A surface with negative curvature.

Problem 2: Compute the first and second and third fundamental form for the torus $r : R \subset \mathbb{R}^2 \to \mathbb{R}^3$

$$r(u,v) = \left[\ (5+\cos(v))\cos(u), (5+\cos(v))\sin(u), \sin(v) \ \right] ,$$

where $u, v \in [0, 2\pi)$. We want you to write down explicit iexpressions for the matrices dr, dn. This problem again can be solved by hand, but you are allowed here to use a computer algebra system to assist you.

Problem 3: a) Compute the matrix A for the shape operator for the same torus.
b) Compute the Gauss curvature K and the mean curvature H.

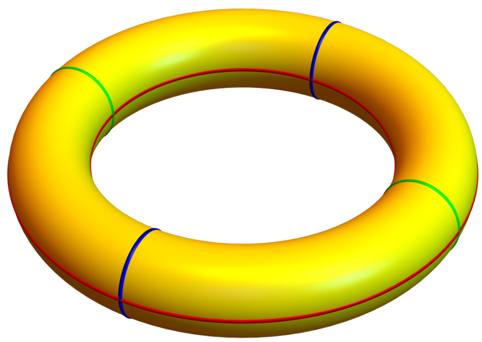

FIGURE 2. The torus for problem 3-4.

Problem 4: a) Again continuing with the torus, integrate $\int_0^{2\pi} \int_0^{2\pi} K dV$. Even if you should use a computer algebra system to integrate, you need to find out what integration method can be used to solve the integral at hand.
b) Your result will be an integer but it will not be compatible with the Gauss Bonnet result that you have seen in unit 9. What was going wrong?

Problem 5: a) Verify that the shape operator matrix A is symmetric in the inner product $\langle v, w \rangle = v^T I w$ so that we can parametrize the surface with a basis such that A is diagonal with respect to the inner product given by I and so that $n_u = -\lambda r_u$ and $n_v = -\mu r_v$. (You assume that I is already diagonal).
b) Use a) to prove the matrix identity $III - 2HII + KI = 0$, where $H = (\lambda + \mu)/2$ and $K = \lambda\mu$ and λ, μ are the eigenvalues of A.

P.S. Here is Mathematica sample code to compute. If you chose to use a computer algebra system, we ask you in problem 1) and 2) to comment what each of the commands you enter does.

```
r={Sin[v] Cos[u],Sin[v] Sin[u],Cos[v]};
ru=D[r,u]; rv=D[r,v];
n=Cross[ru,rv]; n=n/Sqrt[n.n];
nu=D[n,u]; nv=D[n,v];
drt={ru,rv};dr=Transpose[drt];
dnt={nu,nv};dn=Transpose[dnt];
g=drt.dr;   h=-dnt.dr;    e=dnt.dn;
A=Inverse[g].h;    H=Tr[A]/2;   K=Det[A];
```

DIFFERENTIAL GEOMETRY

MATH 136

Unit 10 Homework

This is the sixth homework. It is due Friday, October 25rd.

Problem 1: Compute the curvatures of the Pentakis Icosahedron (Golden Fullerene) and verify Gauss-Bonnet in this case.

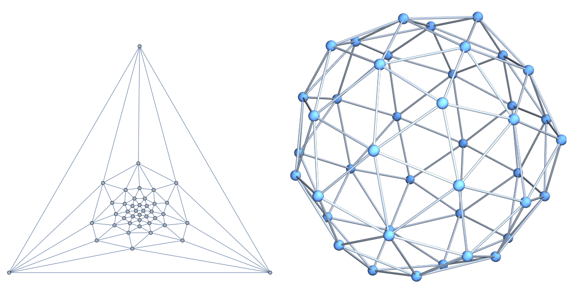

FIGURE 1. The Pentakisicosahedron is a 2-sphere with 42 vertices, 120 edges and 80 triangles. In chemistry it appears as $Au_{30}Si_{12}$.

Problem 2: A graph is called 1-dimensional if it does not contain any triangle. The curvature of a 1-dimensional graph is defined as $1-|S(v)|/2$, where $S(v)$ is the unit sphere. As for 2-manifolds, $|S(v)|$ agrees with the vertex degree $|S_0(v)|$.
a) Along the same line as the Gauss-Bonnet theorem for 2-manifold, prove that $\chi(G) = \sum_{v \in V} K(v)$ for a any 1-dimensional graph.
b) A graph which does not contain any circular subgraph is called a **forest**. A connected component of a forest is called a **tree**. Prove that the Euler characteristic of a forest is equal to the number of trees.
c) Explain why points of positive curvature are called "leaves" and points of negative curvature are "branch points".
d) A "flower" is a circular graph (C_n with $n \geq 4$), where each vertex can be attached a tree. Compute the Euler characteristic of a flower.

Problem 3: A 3-manifold is a finite simple graph for which every unit sphere is a 2-sphere. The classification of 3-manifolds is much more difficult than the classification of 2-manifolds.
a) Look up the 600 cell and the 16 cell and show that they are 3-manifolds.
b) Verify that if H_1, H_2 are 1-spheres, then the join $H_1 \oplus H_2$ obtained by taking the disjoint union of the two graphs and connecting every vertex in H_1 with a vertex in H_2 is a 3-manifold.

Problem 4: A **2-manifold with boundary** is a graph such that every unit sphere is either a circular graph C_n with $n \geq 4$ vertices or a path graph P_n with $n \geq 2$ vertices. The former points are called interior points. The curvature at a general point is $K(v) = 1 - |S_0(v)|/2 + |S_1(v)|/3$, where $S_0(v)$ is the set of vertices of $S(v)$ and $S_1(v)$ is the set of edges in $S(v)$. a) Verify from this definition that for a manifold with boundary, the curvature of a boundary point is $1/2 - |S_1(v)|/6$ and the curvature is $K(v) = 1 - |S_1(v)|/6$ for interior points. b) Check that the graph complement G of C_7 is a 2-manifold without interior. It implements the smallest Möbius strip. Verify that all curvatures of G are zero.

Problem 5: a) The **Barycentric refinement** of a 2-manifold $G = (V, E)$ takes $V' = V \cup E \cup F$ as vertices and takes as E' the set of pairs (x, y) such that $x \subset y$ or $y \subset x$. If $f = [|V|, |E|, |F|]$ is the f-vector of G, then $f(G') = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 6 \end{bmatrix} f(G)$. Conclude that the Euler characteristic is invariant. (Hint: Show that $[1, -1, 1]$ is an eigenvector of A^T.)
b) The **soft Barycentric refinement** of 2-manifold takes $V' = V \cup F$ as vertices and E' as the set of pairs (x, y) such that $x \subset y$ or $y \subset x$ or $x \cap y$ is in E. Now the f-vectors transform as $f' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix} f$. Verify again that the Euler characteristic satisfies $\chi(G) = \chi(G')$.

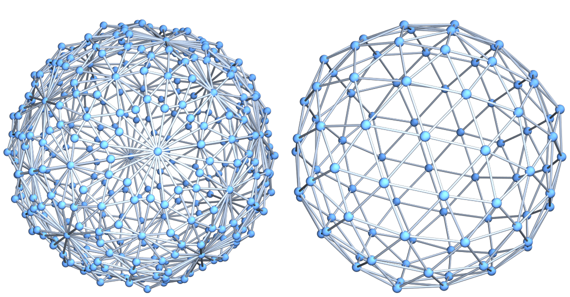

FIGURE 2. The second Barycentric refinement and the second Soft Barycentric refinement of an icosahedron.

DIFFERENTIAL GEOMETRY

MATH 136

Unit 11 Homework

This is the seventh homework. It is due Friday, November 1st:

Problem 1: Assume you make a trip and your effort is $x\dot{x}^2$ rather than the kinetic energy $\dot{x}^2$ because things are easy initially but get harder later on. What is the best strategy to reach from $x(0) = 0$ to $x(1) = 1$? Go slow first or go fast first? In order to find the best way, solve the Euler-Lagrange problem to minimize the action

$$E(x) = \int_0^1 F(x, \dot{x}) \, dt$$

for $F(x, \dot{x}) = x\dot{x}^2$ with $x(0) = 0, x(1) = 1$.

Problem 2: Look at the paraboloid $r(u, v) = [u, v, u^2 + v^2]$.
a) Compute all the Christoffel symbols Γ_{ijk}. These are 8 terms.
b) Now compute all the Christoffel symbols Γ_{ij}^k.

Problem 3: A geodesic $x(t)$ is called **closed**, if there exists L such that $x(L) = x(0)$ and $\dot{x}(L) = \dot{x}(0)$. It is a closed curve on M for which also initial and end velocities agree. Note that the curve x does not have to be simple. Two closed geodesics $x_1(t), x_2(t)$ are called **homotopic**, if one can deform them to each other on the manifold. Formally this means to parametrize both on $[a, b] = [0, 1]$ and then give a continuous $F(t, s)$ of two variables, such that $F(t, 0) = x_1(t)$ and $F(t, 1) = x_2(t)$ and $t \to x_s(t) = F(t, s)$ is on the manifold. We verify here that on any torus, there are infinitely many geodesics that are pairwise not homotopic to each other.
a) Invent and then define a "winding vector" $(n, m) \in \mathbb{Z}^2$.
b) Verify that this number is the same for two closed curves that are homotopic.
c) Draw a torus and a geodesic for winding number $(4, 5)$.
d) Verify that there each homotopy class is not empty by telling how to write one.
e) Argue why there is at least one geodesic in each homotopy class.

Problem 4: We draw some wave fronts $W_r(p)$ on the flat Clifford torus $\mathbb{R}^2/\mathbb{Z}^2$ (Pac-Man square). This manifold can be realized as all point (x, y) in $\mathbb{R}^2$, where points $(x, y), (x + n, x + m)$ identified if $n, m \in \mathbb{Z}$. Work with the point $p = (1/2, 1/2)$.
a) Draw the wave front $W_{1/2}(p)$.
b) Draw the wave front $W_1(p)$.
c) Draw the wave front $W_5(p)$.
You are welcome to become physical part c and use scissor, ruler and compass to do that).

1

Problem 5: The torus

$$r(u, v) = ((a + b\cos(v))\cos(u), (a + b\cos(v))\sin(u), b\sin(v)) \ ,$$

has the metric

$$\begin{aligned} g_{11} &= (a + b\cos(v))^2 \\ g_{22} &= b^2 \\ g_{12} &= g_{21} = 0 \end{aligned}$$

Use the following example code (done for the sphere) to compute all the Christoffel symbols Γ^k_{ij}. Make sure to simplify.

```
r={Sin[v] Cos[u],Sin[v] Sin[u],Cos[v]};
ru=D[r,u];            rv=D[r,v];
n=Cross[ru,rv];  n=n/Sqrt[n.n];
nu=D[n,u];            nv=D[n,v];
drT={ru,rv};          dr=Transpose[drT];
g=drT.dr;             gi=Inverse[g];
dnT={nu,nv};          dn=Transpose[dnT];
h=-dnT.dr;            e=dnT.dn;
K=Det[h]/Det[g];
X={u,v};  d=2;
c[i_,j_,k_]:=(D[g[[j,k]],X[[i]]]
              +D[g[[k,i]],X[[j]]]
              -D[g[[i,j]],X[[k]]])/2;
Christoffel[i_,j_,k_]:=Sum[gi[[k,l]]*c[i,j,l],{l,d}];
S=Table[Simplify[Christoffel[i,j,k]],{i,d},{j,d},{k,d}];
TableForm[S]
```

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[1]The wave fronts on a torus become dense as shown in a project of Emily Kang of Summer 2024.

DIFFERENTIAL GEOMETRY

MATH 136

Unit 12-13 Homework

This is the eighth homework. It is due Friday, November 8st:

Problem 13.1: To warm up to Greens theorem. Solve the following problem: use Green to compute the area of the region $|x|^{2/3}/a^2 + |y|^{2/3}/b^2 \leq 1$.

Problem 13.2: a) Green's theorem tells that if $R \subset \mathbb{R}^2$ is a region and $X = [P, Q]$ is a vector field in the plane, then $\iint_R \text{curl} X \; dudv = \int_{\delta R} X(r(t)) \cdot r'(t) \; dt$ where δR is the boundary. Look up and write down a proof of this.
b) Look up the discrete Green theorem and give a proof

[1]

Problem 13.3: Verify here that Stokes theorem on $S = r(R)$ can be reduced to Green on R:

$$\iint_R \text{curl}(F) r_u \times r_v \; dudv = \iint_R \text{curl}(X) dudv$$

Assume $F = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ is a vector field in space. Prove the important formula

$$\text{curl}(F) \cdot r_u \times r_v = F_u \cdot r_v - F_v \cdot r_u \; .$$

As we have seen in class, this implies that the 2D field $X = [F.r_u, F.r_v]$ satisfies $\text{curl}(X) = F_u \cdot r_v - F_v \cdot r_u$.

Problem 13.4: We have seen half of the proof that the form X is intrinsic. Verify that also the second part of $X = [z \cdot w_u, z \cdot w_v]$ can be expressed from I alone.

[1] https://people.math.harvard.edu/ knill/teaching/math22b2022/handouts/lecture33.pdf

Problem 13.5: Below you see Gauss's original statement of the theorema Egregium translated into English. Explain what he means with "developing a surface upon any other surface" and why it is not possible for example to find a map of the earth in $\mathbb{R}^3$ which preserves distances.

Suppose that our surface can be developed upon another surface, curved or plane, so that to each point of the former surface, determined by the coordinates x, y, z, will correspond a definite point of the latter surface, whose coordinates are x', y', z'. Evidently x', y', z' can also be regarded as functions of the indeterminates p, q, and therefore for the element $\sqrt{(dx'^2 + dy'^2 + dz'^2)}$ we shall have an expression of the form

$$\sqrt{(E'\,dp^2 + 2F'\,dp\,.\,dq + G'\,dq^2)}$$

where E', F', G' also denote functions of p, q. But from the very notion of the *development* of one surface upon another it is clear that the elements corresponding to one another on the two surfaces are necessarily equal. Therefore we shall have identically

$$E = E', \quad F = F', \quad G = G'.$$

Thus the formula of the preceding article leads of itself to the remarkable

THEOREM. *If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged.*

FIGURE 1. Gauss original statement translated into English. Source: Wikipedia

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DIFFERENTIAL GEOMETRY

MATH 136

Unit 15-16 Homework

This is the ninth' homework. It is due Friday, November 15th:

Problem 1: a) Prove the **Euler Handshake lemma** $\sum_{i=1}^{V} d_i = 2E$ in graph theory.
b) Show that for a discrete 2-manifold with F faces and Euler characteristic $\chi(M)$ the number V of vertices V and the number E of edges E are determined.
c) In lecture 16, you see a picture of a discrete torus with $V = 64$ vertices. Determine E and F in that case.

Problem 2: We parametrize a paraboloid M as $r(u, v) = (u, v, u^2 + v^2)$. for $R = \{u^2 + v^2 \leq 1\}$. This is a **2-manifold with boundary**.
a) Compute the curvature K.
b) Compute $|r_u \times r_v| = \sqrt{\det(g)}$.
c) Compute $\iint_R K \, dV$.

Problem 3: We continue with the same paraboloid as before.
a) Compute the curvature of the boundary curve $x(t)$ (parametrized by arc length).
b) Compute the normal curvature $\kappa_n(t) = n(t) \cdot \ddot{x}(t)$ as well as the geodesic curvature $\kappa_g(t) = (n \times \dot{x}) \cdot \ddot{x}$.
c) Verify the local Gauss-Bonnet result. That is show that

$$\iint_R K \, dV + \int_0^L \kappa_g(t) \, dt = 2\pi \ .$$

Problem 4: Use a computer algebra system to verify that

$$\iint_M K \, dV = 4\pi$$

if $M = \{x^2/a^2 + y^2/b^2 + z^2/c^2 = 1\}$ for $a = 2, b = 3, c = 5$. What is the maximal and what is the minimal curvature of this ellipsoid?

Problem 5: The "angular defect" $K(p)$ at a vertex of a convex polyhedron M is the angle needed to add to complete the angle to 2π. For a cube for example, it $2\pi - 3\pi/2 = \pi/2$ at every corner.
a) Descartes theorem states that the total defect of a convex polyhedron is 4π so that the angular defect is a curvature. This is a polyhedral Gauss-Bonnet theorem. Verify this for an icosahedron to see what is going on.
b) Verify that for a general polyhedral surface $K(p) = 2\pi - \sum_i \alpha_i$ gives a curvature that adds up to 2π times the Euler characteristic of the surface. Here, α_i are the angle interior angles and the result you want to show is $\sum_p K(p) = 2\pi\chi(M)$. It is a version of Gauss-Bonnet.
c) Illustrate your theorem with the Escher stair polyhedron built in **mine craft** or **Lego**. Compute all the angular defects and add them up. The total curvature should be the Euler characteristic of the stair.

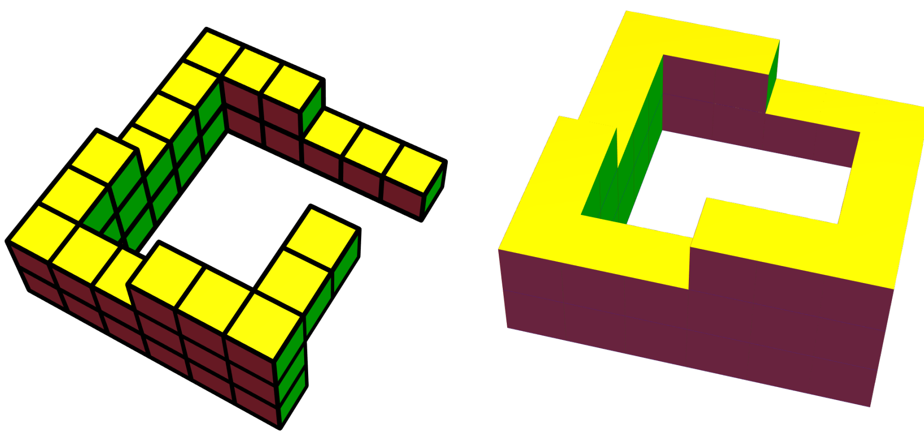

Figure 1. The Escher Stairs built in mine craft. If you look at it from the right angle and do glue the bricks nicely, you see an impossible stair, which always goes down or up depending on whether you are a "wineglass half empty" or "wine glass half full" type of person.

DIFFERENTIAL GEOMETRY

MATH 136

Unit 17-18 Homework

This is the 10th homework. It is due Friday, November 22th:

Problem 1: a) Give a definition of what a **Riemannian C^k-manifold with boundary** is. There are very little changes needed to the definition given in the course. You just need to use besides $\mathbb{R}^m$ also $H^m = \{x \in \mathbb{R}^m, x_1 \geq 0\}$.
b) Conclude from the definition that the boundary is again a C^k manifold by giving a concrete atlas for this manifold.

Problem 2: From each of the following objects determine whether it is a tensor field or not
a) The calculus gradient field $\nabla f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$.
b) The Jacobian of a function $df = [f_x, f_y, f_z]$
c) The second fundamental form II
d) A linear transformation field of TM like the shape operator A like $(Av)^k = \sum_j A^k_j v^j$.
e) The inverse I^{-1} of the first fundamental form g^{ij}.
f) The second derivative $r_{u^i u^j}$.
g) The Christoffel symbols $\Gamma_{ijk} = r_{u^i u^j} r_{u^k}$.

Problem 3: Prove that $SU(2)$ is a manifold by explicitly giving the charts.

Problem 4: Draw a 2-dimensional discrete manifold M of your choice by hand. You can find manifolds which are planar, meaning that they can be realized in the plane without crossings of the edges. Also chose a random function f on the vertices taking values in $\{0, 1\}$. Now look at the level curve M_f as defined in the text.

Problem 5: The following code allows you to experiment with level sets in discrete manifolds. The host manifold is a discrete 4-manifold, the join of a 2-sphere and a 1-sphere. Running the code builds a random function from the vertex list to $\{1, 2, 3\}$. It defines a co-dimension 2 manifold.
a) Run the code as it is, report the numbers V,E,F as well as the Euler characteristic of M_f.
b) Change the code and see what happens if the function takes 4 values rather than 3.
c) Build a 5 manifold as the join of two 2-manifolds and build a 3-manifold by taking a function taking 3 random values.
d) Report the curvature values of your 3-manifold.
e) Report the curvature vales of a 4 manifold by taking a function taking 2 values on the 5 manifold in c).

```
Generate[A_]:=If[A=={},{},Sort[Delete[Union[Sort[Flatten[Map[Subsets,A],1]]],1]]];
Whitney[s_]:=Generate[FindClique[s,Infinity,All]]; w[x_]:=-(-1)^k;
R[G_,k_]:=Module[{},R[x_]:=x->RandomChoice[Range[k]]; Map[R,Union[Flatten[G]]]];
F[G_]:=Delete[BinCounts[Map[Length,G]],1]; Euler[G_]:=F[G].Table[w[k],{k,Length[F[G]]}];
Surface[G_,g_]:=Select[G,SubsetQ[#/.g,Union[Flatten[G] /. g]] &];
S[s_,v_]:=VertexDelete[NeighborhoodGraph[s,v],v];       Sf[s_,v_]:=F[Whitney[S[s,v]]];
Curvature[s_,v_]:=Module[{f=Sf[s,v]},1+f.Table[(-1)^k/(k+1),{k,Length[f]}]];
Curvatures[s_]:=Module[{V=VertexList[s]},Table[Curvature[s,V[[k]]],{k,Length[V]}]];
J[G_,H_]:=Union[G,H+Max[G]+1,Map[Flatten,Map[Union,Flatten[Tuples[{G,H+Max[G]+1}],0]]]];
ToGraph[G_]:=UndirectedGraph[n=Length[G];Graph[Range[n],
  Select[Flatten[Table[k->l,{k,n},{l,k+1,n}],1],(SubsetQ[G[[#[[2]]]],G[[#[[1]]]]])&]]];
Barycentric[s_]:=ToGraph[Whitney[s]];

G=J[Whitney[Barycentric[CompleteGraph[{2,2,2}]]],Whitney[CycleGraph[7]]];   (* J=Join *)
g=R[G,3]; H=Surface[G,g];     (* A codimension 2 manifold in the 4-sphere G=Oct * C_7  *)
Print["EulerChi=",Euler[H]]; Print["Fvector:",F[H]]; s=ToGraph[H]; GraphPlot3D[s]
Print["Gauss-Bonnet Check:"]; Print[Total[Curvatures[s]]==Euler[H]];
Print["Curvature Values:"];   Print[Union[Curvatures[s]]];
```

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DIFFERENTIAL GEOMETRY

MATH 136

Unit 19-20 Homework

Fünf Variationen

You verify a few things using computer algebra. Instructions are on the website. These problems are five variations, analog to a common theme in music (Mozart (K. 501) or Beethoven on 'Rule Britannia" (WoO 79)).

Problem 1: Verify that the Riemannian manifold (M, g) defined by the parametrization

$$r(u,v) = [(5+2\cos(v))\cos(u), (5+2\sin(v))\sin(u), \cos(v)]$$

satisfies the Einstein equations

$$R - \frac{1}{2}Sg = 0$$

where R is the Ricci tensor, where S is the scalar curvature and where g is the first fundamental form aka Riemannian metric. What is the scalar curvature S for $v = \pi/2$?

Problem 2: Verify that the Schwarzschild metric $(M = \mathbb{R}^4, g)$ with

$$g = \begin{bmatrix} \frac{2M}{r} - 1 & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2M}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2(\phi) \end{bmatrix}$$

satisfies the Einstein equations

$$R - \frac{1}{2}Sg = 0$$

What is the Riemann tensor entry R_{1122} for $r = 1, M = 1$?

Problem 3: The Reissner-Nordstroem metric

$$g = \begin{bmatrix} -\frac{e^2}{r^2} + \frac{2M}{r} - 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\frac{e^2}{r^2} - \frac{2M}{r} + 1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2(\phi) \end{bmatrix}$$

is a static solution of the Einstein-Maxwell equations

$$R - \frac{1}{2} S g = T \, .$$

But it is not a vacuum. It describes the field of a charged, non-rotating body of mass M and charge e. An example is a charged black hole. What is the entry T_{44} for $e = 1, r = 1, \phi = \pi/3$?

Problem 4: Verify that the metric on $SU(2) = S^3$ given by the parametrization

$$r = [\cos(u)\cos(w), \sin(u)\cos(w), \cos(v)\sin(w), \sin(v)\sin(w)]$$

satisfies vacuum Einstein equations $R - 2g = 0$. What is S?

Problem 5: Finally check that the pseudo sphere given by the parametrization

$$r = [\cos(u)\sin(v), \sin(u)\sin(v), \cos(v) + \log\left(\tan(\frac{v}{2})\right)]$$

satisfies the vacuum Einstein equations. What is the curvature K?

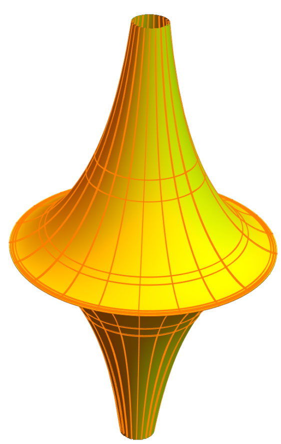

Figure 1. The Pseudo sphere

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