DIFFERENTIAL GEOMETRY

MATH 136

Unit 1: Homework

This is the first homework. It is due Friday September 13.

Viviani's curve

Problem 1.1: The curve

$$r(t) = \begin{bmatrix} \cos^2(t) - \frac{1}{2} \\ \sin(t)\cos(t) \\ \sin(t) \end{bmatrix}$$

is called **Viviani's curve**. Verify that this curve is on the intersection of the sphere $f_1(x,y,z) = (x+1/2)^2 + y^2 + z^2 = 1$ and the cylinder $f_2(x,y) = x^2 + y^2 = 1/4$.

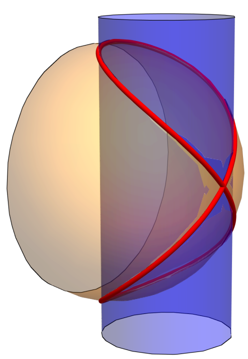

Figure 1. The Viviani curve is the intersection of a sphere and a cylinder.

Problem 1.2: Compute the velocity $r'(t)$ and acceleration $r''(t)$ as well as the jerk $r'''(t)$.
a) Compute the **curvature** of the curve at $t = 0$.
b) Compute the **torsion** of the curve at $t = 0$.

Problem 1.3: Lets look at the map from $\mathbb{R}^3$ to $\mathbb{R}^2$ given by

$$f(x,y,z) = \begin{bmatrix} f_1(x,y,z) \\ f_2(x,y,z) \end{bmatrix} .$$

Viviani's curve can be written as the set $f(x,y,z) = c$ with $c = \begin{bmatrix} 1 \\ 1/4 \end{bmatrix}$. Compute the **Jacobean matrix**

$$df(x,y,z) = \begin{bmatrix} \frac{\partial}{\partial x} f_1 & \frac{\partial}{\partial y} f_1 & \frac{\partial}{\partial z} f_1 \\ \frac{\partial}{\partial x} f_2 & \frac{\partial}{\partial y} f_2 & \frac{\partial}{\partial z} f_2 \end{bmatrix} .$$

Problem 1.4: Viviani's curve is the intersection of two polynomial equations. It is known as an **algebraic curve**. It is not a manifold however. With a crossing, it is topologically a **figure 8**. By the implicit function theorem, the curve is **manifold like** near any point, where the Jacobean matrix has rank 2 (which in our case means that the gradients are not parallel). Verify that the matrix df has rank 1 at $(1/2, 0, 0)$ and rank 2 everywhere else.

Problem 1.5: a) Lets compute the curvature of a sphere of radius 1 using the Puiseux's formula. The sphere is parametrized as

$$r(u,v) = [\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)] .$$

Let p be a point on the sphere (take the north pole) and $S_r(p)$ the sphere of radius r centered at p. First verify that in the north pole case $S_r(p)$ is parametrized as $r(t) = [\cos(t)\sin(r), \sin(t)\sin(r), \cos(r)]$. Now check that $K(p) = \lim_{r \to 0} 3\frac{2\pi r - |S_r(p)|)}{\pi r^3} = 1$.
b) Use the same curvature notion to compute it for the cylinder of radius 1/4 appearing in Vivianis's curve.

Remark: Much of our course will be devoted to curvature. We will in the third week look at curvature and torsion and see that they determine a curve up to translation. Later in the course we look at curvature of surfaces defined as $K = \det(II)/\det(I)$ using matrices I, II called first and second fundamental form. This will lead to the Gauss-Bonnet formula. The Puiseux formula for curvature is an intuitive notion of curvature which we will be able to show to be equivalent to the "official curvature". But that is far from obvious. The statement that curvature is an intrinsic notion of a surface and does not depend on any embedding of the surface in space is the "Theorema Egregium". It will be one of the goals of this course to understand this.

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