DIFFERENTIAL GEOMETRY

MATH 136

Unit 4-5 Homework

This is the third homework. It is due Friday, September 27.

Problem 1: Almost all books or courses give as problem number 1 in the course the task to prove the formulas

$$\kappa = |r' \times r''|/|r'|^3, \quad \tau = (r' \times r'') \cdot r'''/|r' \times r''|^2 .$$

We actually will prove these formulas in class. You do not have to reprove them here. What we want you do, is the much easier reverse: give a detailed proof that if $r(t)$ is parametrized by arc length, then these formulas agree with the formulas for curvature and torsion you have seen in class, that is the formulas which define curvature and torsion in the case of constant arc length parametrization.

Problem 2: a) Look up and write down a proof that if $F(t,x)$ is a differentiable function from $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, and $x_0 \in \mathbb{R}^n$, then there exists an open interval $(-a,a)$ and a unique path $x(t) : (-a,a) \to \mathbb{R}^N$ such that $x' = F(t,x)$ and $x(0) = x_0$. We want you to write down the proof in the differentiable case which is a bit more special than the usual assumption assuming a Lipschitz property for F.
b) Justify that if $x(t)$ stays bounded meaning that there is constant such that $|x(t)| \leq M$ for all t, then the solution exists for all t. (We call this a global solution.) Now conclude that if $Q' = K(t)Q$ is a differential equation for a matrix $Q(t)$ with skew symmetric $K(t)$, then there is a global solution.

Problem 3: a) Determine from each of the spaces $SO(n), so(n), SU(n), su(n)$ whether they are linear spaces or not.
b) Check that if $x(t)$ is a differentiable curve in $SO(n)$, then $x(t)$ satisfies the differential equation $x'(t) = A(t)x(t)$, where $A(t) \in so(n)$, the space of skew-symmetric matrices.
c) Show that $A(t) = A$ is a constant skew symmetric matrix, then the **matrix exponential** $Q(t) = e^{At}$ is an orthogonal matrix. What is this matrix $Q(t)$ in the case $n = 2$?

Problem 4: a) First verify that the helix $r(t) = [\cos(at), \sin(at), bt]$ has constant curvature and torsion. What are the values?
b) Now prove that if a curve has constant curvature and torsion, it must be a helix.

Problem 5: a) Verify that if $B(t) \in so(n)$ and $Q' = BQ$, then $L(t) = Q(t)L(0)Q^T(t)$ satisfies the so called **Lax pair** differential equation

$$L' = [B, L] = BL - LB \; .$$

Conclude that the eigenvalues of L are preserved. (This fact an important part of the theory of **integrable systems** which can explain phenomena of solitons.)
b) (*) Let $B(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Write down the Lax pair differential equation and solve it for $L(0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(*) This is probably the simplest non-trivial example of a Lax pair. It had been given by Hermann Flaschka (1945-2021), one of the pioneers in integrable systems in 1974. Flaschka had been the chair of the math department in Tucson when I (Oliver) had been teaching there.

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