DIFFERENTIAL GEOMETRY

MATH 136

Unit 6-7 Homework

This is the fourth homework. It is due Friday, October 4rd.

Problem 1: Check the Hopf Umlaufsatz in the example of an asteriod

$$r(t) = [\cos^3(t), \sin^3(t)] \ .$$

First verify that $|r'(t)| = |3\cos(t)\sin(t)|$ and $\kappa(t) = -(2/3)/|\sin(2t)|$ then compute the rotation index $\int_0^{2\pi} \kappa(t)|r'(t)| \, dt/(2\pi)$. While your result will comply with the Hopf Umlaufsatz, there is something strange going on given how you rotate counterclockwise around the region. Figure it out!

Problem 2: a) Compute the rotation index in the case of the simple closed curve

$$r(t) = 17[\cos(t), \sin(t)] - [\cos(17t), \sin(17t)] \ .$$

You will see that $\int_0^{2\pi} \kappa(t)|r'(t)| \, dt/(2\pi) = 9$. We have complete melt-down of the Umlaufsatz. Comment on what is going on.
b) Now do the computation when 17 is replaced by 2:

$$r(t) = 2[\cos(t), \sin(t)] - [\cos(2t), \sin(2t)] \ .$$

You will get the rotation index 3/2. **W**hy **T**his **F**ailure?

Problem 3: A **discrete Hopf Umlaufsatz** can be formulated for polygons with n vertices.
a) Assume first we have a simple convex polygon with n vertices. Define the curvature at the vertex v_k to be κ_k which is the outer angle $\pi - \alpha_k$ where α_k is the angle you have defined in third grade for polygons. The discrete Hopf Umlaufsatz tells $\sum_{k=1}^n \kappa_k = 2\pi$. Prove this.
b) Now formulate the general (not necessarily convex) case. Define suitable curvatures such that the result works.

Problem 4: Check the four vertex theorem in the example

$$r(t) = 4[\cos(t), \sin(t)] - [\cos(2t), \sin(2t)] \ .$$

The expressions for $\kappa(t)$ and $\kappa'(t)$ are not that bad. Plot the function $\kappa(t)$ and find the critical points, the roots of κ'.

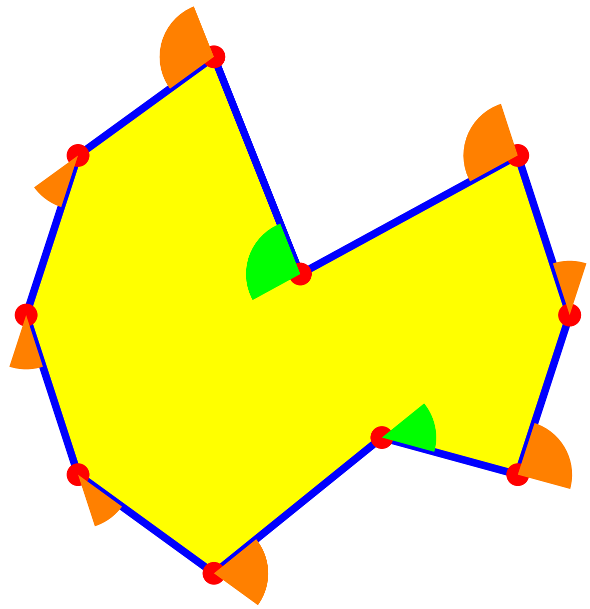

FIGURE 1. Problem 2: Proving the Umlaufsatz for polygons. Curvatures can get positive or negative.

Problem 5: For parameters a, define $c = 2\sqrt{(a)}$ and the curve $r(t) = [a\cos(t)+\cos(3t), a\sin(t)-\sin(3t), c\sin(2t)]$. For $a = 1$ one has the **tennis ball curve**, for $a = 1/2$ the **base ball curve** and for $a = 1.8$, the **basket ball curve** (Basketballs have two additional grand circles).
a) Verify that these curves are located on a sphere.
b) Look up the tennisball theorem, state its content, then write down the main idea on how the theorem is proven.

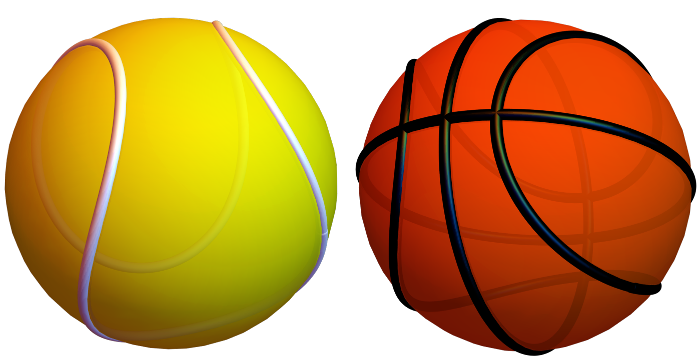

FIGURE 2. Problem 5: A tennis ball and a basket ball.