**DIFFERENTIAL GEOMETRY**

MATH 136

# Unit 10 Homework

This is the sixth homework. It is due Friday, October 25rd.

**Problem 1:** Compute the curvatures of the Pentakis Icosahedron (Golden Fullerene) and verify Gauss-Bonnet in this case.

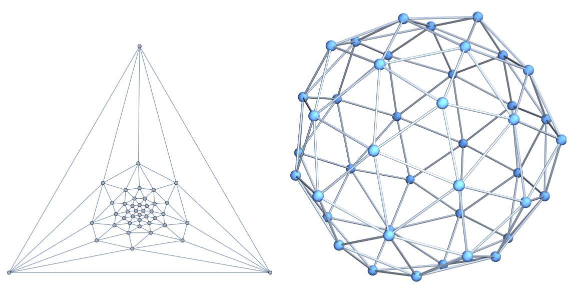

FIGURE 1. The Pentakisicosahedron is a 2-sphere with 42 vertices, 120 edges and 80 triangles. In chemistry it appears as $Au_{30}Si_{12}$.

**Problem 2:** A graph is called 1-dimensional if it does not contain any triangle. The curvature of a 1-dimensional graph is defined as $1-|S(v)|/2$, where $S(v)$ is the unit sphere. As for 2-manifolds, $|S(v)|$ agrees with the vertex degree $|S_0(v)|$.
a) Along the same line as the Gauss-Bonnet theorem for 2-manifold, prove that $\chi(G) = \sum_{v \in V} K(v)$ for a any 1-dimensional graph.
b) A graph which does not contain any circular subgraph is called a **forest**. A connected component of a forest is called a **tree**. Prove that the Euler characteristic of a forest is equal to the number of trees.
c) Explain why points of positive curvature are called "leaves" and points of negative curvature are "branch points".
d) A "flower" is a circular graph ($C_n$ with $n \geq 4$), where each vertex can be attached a tree. Compute the Euler characteristic of a flower.

**Problem 3:** A 3-manifold is a finite simple graph for which every unit sphere is a 2-sphere. The classification of 3-manifolds is much more difficult than the classification of 2-manifolds.
a) Look up the 600 cell and the 16 cell and show that they are 3-manifolds.
b) Verify that if $H_1, H_2$ are 1-spheres, then the join $H_1 \oplus H_2$ obtained by taking the disjoint union of the two graphs and connecting every vertex in $H_1$ with a vertex in $H_2$ is a 3-manifold.

**Problem 4:** A **2-manifold with boundary** is a graph such that every unit sphere is either a circular graph $C_n$ with $n \geq 4$ vertices or a path graph $P_n$ with $n \geq 2$ vertices. The former points are called interior points. The curvature at a general point is $K(v) = 1 - |S_0(v)|/2 + |S_1(v)|/3$, where $S_0(v)$ is the set of vertices of $S(v)$ and $S_1(v)$ is the set of edges in $S(v)$. a) Verify from this definition that for a manifold with boundary, the curvature of a boundary point is $1/2 - |S_1(v)|/6$ and the curvature is $K(v) = 1 - |S_1(v)|/6$ for interior points. b) Check that the graph complement $G$ of $C_7$ is a 2-manifold without interior. It implements the smallest Möbius strip. Verify that all curvatures of $G$ are zero.

**Problem 5:** a) The **Barycentric refinement** of a 2-manifold $G = (V, E)$ takes $V' = V \cup E \cup F$ as vertices and takes as $E'$ the set of pairs $(x, y)$ such that $x \subset y$ or $y \subset x$. If $f = [|V|, |E|, |F|]$ is the $f$-vector of $G$, then $f(G') = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 6 \\ 0 & 0 & 6 \end{bmatrix} f(G)$. Conclude that the Euler characteristic is invariant. (Hint: Show that $[1, -1, 1]$ is an eigenvector of $A^T$.)
b) The **soft Barycentric refinement** of 2-manifold takes $V' = V \cup F$ as vertices and $E'$ as the set of pairs $(x, y)$ such that $x \subset y$ or $y \subset x$ or $x \cap y$ is in $E$. Now the $f$-vectors transform as $f' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix} f$. Verify again that the Euler characteristic satisfies $\chi(G) = \chi(G')$.

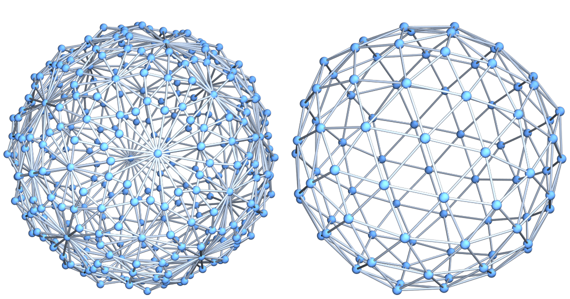

FIGURE 2. The second Barycentric refinement and the second Soft Barycentric refinement of an icosahedron.