

DIFFERENTIAL GEOMETRY

MATH 136

Unit 11 Homework

This is the seventh homework. It is due Friday, November 1st:

Problem 1: Assume you make a trip and your effort is $x\dot{x}^2$ rather than the kinetic energy \dot{x}^2 because things are easy initially but get harder later on. What is the best strategy to reach from $x(0) = 0$ to $x(1) = 1$? Go slow first or go fast first? In order to find the best way, solve the Euler-Lagrange problem to minimize the action

$$E(x) = \int_0^1 F(x, \dot{x}) dt$$

for $F(x, \dot{x}) = x\dot{x}^2$ with $x(0) = 0, x(1) = 1$.

Problem 2: Look at the paraboloid $r(u, v) = [u, v, u^2 + v^2]$.

- Compute all the Christoffel symbols Γ_{ijk} . These are 8 terms.
- Now compute all the Christoffel symbols Γ_{ij}^k .

Problem 3: A geodesic $x(t)$ is called **closed**, if there exists L such that $x(L) = x(0)$ and $\dot{x}(L) = \dot{x}(0)$. It is a closed curve on M for which also initial and end velocities agree. Note that the curve x does not have to be simple. Two closed geodesics $x_1(t), x_2(t)$ are called **homotopic**, if one can deform them to each other on the manifold. Formally this means to parametrize both on $[a, b] = [0, 1]$ and then give a continuous $F(t, s)$ of two variables, such that $F(t, 0) = x_1(t)$ and $F(t, 1) = x_2(t)$ and $t \rightarrow x_s(t) = F(t, s)$ is on the manifold. We verify here that on any torus, there are infinitely many geodesics that are pairwise not homotopic to each other.

- Invent and then define a “winding vector” $(n, m) \in \mathbb{Z}^2$.
- Verify that this number is the same for two closed curves that are homotopic.
- Draw a torus and a geodesic for winding number $(4, 5)$.
- Verify that there each homotopy class is not empty by telling how to write one.
- Argue why there is at least one geodesic in each homotopy class.

Problem 4: We draw some wave fronts $W_r(p)$ on the flat Clifford torus $\mathbb{R}^2/\mathbb{Z}^2$ (Pac-Man square). This manifold can be realized as all point (x, y) in \mathbb{R}^2 , where points $(x, y), (x + n, x + m)$ identified if $n, m \in \mathbb{Z}$. Work with the point $p = (1/2, 1/2)$.

- a) Draw the wave front $W_{1/2}(p)$.
- b) Draw the wave front $W_1(p)$.
- c) Draw the wave front $W_5(p)$.

You are welcome to become physical part c and use scissor, ruler and compass to do that).

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Problem 5: The torus

$$r(u, v) = ((a + b \cos(v)) \cos(u), (a + b \cos(v)) \sin(u), b \sin(v)) ,$$

has the metric

$$\begin{aligned} g_{11} &= (a + b \cos(v))^2 \\ g_{22} &= b^2 \\ g_{12} &= g_{21} = 0 \end{aligned}$$

Use the following example code (done for the sphere) to compute all the Christoffel symbols Γ_{ij}^k . Make sure to simplify.

```
r={Sin[v] Cos[u], Sin[v] Sin[u], Cos[v]};
ru=D[r,u]; rv=D[r,v];
n=Cross[ru,rv]; n=n/Sqrt[n.n];
nu=D[n,u]; nv=D[n,v];
drT={ru,rv}; dr=Transpose[drT];
g=drT.dr; gi=Inverse[g];
dnT={nu,nv}; dn=Transpose[dnT];
h=-dnT.dr; e=dnT.dn;
K=Det[h]/Det[g];
X={u,v}; d=2;
c[i_ ,j_ ,k_]:= (D[g[[j,k]],X[[i]]]
+D[g[[k,i]],X[[j]]]
-D[g[[i,j]],X[[k]]])/2;
Christoffel[i_ ,j_ ,k_]:=Sum[gi[[k,l]]*c[i,j,l],{l,d}];
S=Table[Simplify[Christoffel[i,j,k]],{i,d},{j,d},{k,d}];
TableForm[S]
```

¹The wave fronts on a torus become dense as shown in a project of Emily Kang of Summer 2024.