DIFFERENTIAL GEOMETRY

MATH 136

Unit 12-13 Homework

This is the eighth homework. It is due Friday, November 8st:

Problem 13.1: To warm up to Greens theorem. Solve the following problem: use Green to compute the area of the region $|x|^{2/3}/a^2 + |y|^{2/3}/b^2 \leq 1$.

Problem 13.2: a) Green's theorem tells that if $R \subset \mathbb{R}^2$ is a region and $X = [P, Q]$ is a vector field in the plane, then $\iint_R \text{curl} X \; dudv = \int_{\delta R} X(r(t)) \cdot r'(t) \; dt$ where δR is the boundary. Look up and write down a proof of this.
b) Look up the discrete Green theorem and give a proof

[1]

Problem 13.3: Verify here that Stokes theorem on $S = r(R)$ can be reduced to Green on R:

$$\iint_R \text{curl}(F) r_u \times r_v \; dudv = \iint_R \text{curl}(X) dudv$$

Assume $F = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$ is a vector field in space. Prove the important formula

$$\text{curl}(F) \cdot r_u \times r_v = F_u \cdot r_v - F_v \cdot r_u \; .$$

As we have seen in class, this implies that the 2D field $X = [F.r_u, F.r_v]$ satisfies $\text{curl}(X) = F_u \cdot r_v - F_v \cdot r_u$.

Problem 13.4: We have seen half of the proof that the form X is intrinsic. Verify that also the second part of $X = [z \cdot w_u, z \cdot w_v]$ can be expressed from I alone.

[1] https://people.math.harvard.edu/ knill/teaching/math22b2022/handouts/lecture33.pdf

Problem 13.5: Below you see Gauss's original statement of the theorema Egregium translated into English. Explain what he means with "developing a surface upon any other surface" and why it is not possible for example to find a map of the earth in $\mathbb{R}^3$ which preserves distances.

Suppose that our surface can be developed upon another surface, curved or plane, so that to each point of the former surface, determined by the coordinates x, y, z, will correspond a definite point of the latter surface, whose coordinates are x', y', z'. Evidently x', y', z' can also be regarded as functions of the indeterminates p, q, and therefore for the element $\sqrt{(dx'^2 + dy'^2 + dz'^2)}$ we shall have an expression of the form

$$\sqrt{(E' dp^2 + 2F' dp \,.\, dq + G' dq^2)}$$

where E', F', G' also denote functions of p, q. But from the very notion of the *development* of one surface upon another it is clear that the elements corresponding to one another on the two surfaces are necessarily equal. Therefore we shall have identically

$$E = E', \quad F = F', \quad G = G'.$$

Thus the formula of the preceding article leads of itself to the remarkable

THEOREM. *If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged.*

FIGURE 1. Gauss original statement translated into English. Source: Wikipedia

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