

DIFFERENTIAL GEOMETRY

MATH 136

Unit 19-20 Homework

FÜNF VARIATIONEN

You verify a few things using computer algebra. Instructions are on the website. These problems are five variations, analog to a common theme in music (Mozart (K. 501) or Beethoven on ‘Rule Britannia’ (WoO 79)).

Problem 1: Verify that the Riemannian manifold (M, g) defined by the parametrization

$$r(u, v) = [(5 + 2 \cos(v)) \cos(u), (5 + 2 \sin(v)) \sin(u), \cos(v)]$$

satisfies the Einstein equations

$$R - \frac{1}{2}Sg = 0$$

where R is the Ricci tensor, where S is the scalar curvature and where g is the first fundamental form aka Riemannian metric. What is the scalar curvature S for $v = \pi/2$?

Problem 2: Verify that the Schwarzschild metric $(M = \mathbb{R}^4, g)$ with

$$g = \begin{bmatrix} \frac{2M}{r} - 1 & 0 & 0 & 0 \\ 0 & \frac{1}{1 - \frac{2M}{r}} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2(\phi) \end{bmatrix}$$

satisfies the Einstein equations

$$R - \frac{1}{2}Sg = 0$$

What is the Riemann tensor entry R_{1122} for $r = 1, M = 1$?

Problem 3: The Reissner-Nordstrom metric

$$g = \begin{bmatrix} -\frac{e^2}{r^2} + \frac{2M}{r} - 1 & 0 & 0 & 0 \\ 0 & \frac{1}{r^2 - \frac{2M}{r} + 1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2(\phi) \end{bmatrix}$$

is a static solution of the Einstein-Maxwell equations

$$R - \frac{1}{2}Sg = T .$$

But it is not a vacuum. It describes the field of a charged, non-rotating body of mass M and charge e . An example is a charged black hole. What is the entry T_{44} for $e = 1, r = 1, \phi = \pi/3$?

Problem 4: Verify that the metric on $SU(2) = S^3$ given by the parametrization

$$r = [\cos(u) \cos(w), \sin(u) \cos(w), \cos(v) \sin(w), \sin(v) \sin(w)]$$

satisfies vacuum Einstein equations $R - 2g = 0$. What is S ?

Problem 5: Finally check that the pseudo sphere given by the parametrization

$$r = [\cos(u) \sin(v), \sin(u) \sin(v), \cos(v) + \log\left(\tan\left(\frac{v}{2}\right)\right)]$$

satisfies the vacuum Einstein equations. What is the curvature K ?

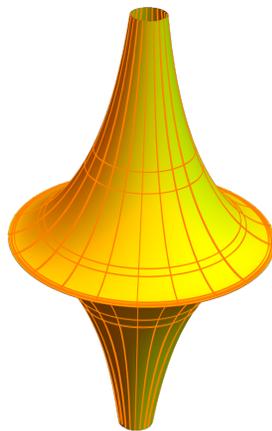


FIGURE 1. The Pseudo sphere