DIFFERENTIAL GEOMETRY

MATH 136

Unit 0: Introduction

0.1. This is an introduction to the Riemannian geometry of curves, surfaces and manifolds. We will also develop some less technical discrete differential geometry and see how the mathematics is applied to general relativity. Most differential geometry texts go too far for a 20 lecture course in which the goal is to prove Gauss-Bonnet in full detail using a multi-variable calculus and linear algebra background only.

0.2. Many texts in Riemannian geometry already early try to go “coordinate free”. This is more elegant but also more abstract. It can come with frustrations. How can one “see” or “work” with an axiomatically defined connection for example? We opted for a concrete approach in which everything can be computed explicitly, either by hand or by using a few lines of a symbolic algebra. A few lines are needed only to compute the Einstein tensor of a manifold. Geodesics are directly introduced using the Euler-Lagrange variational principle. Code provided to visualize these geodesics in an arbitrary given manifold.

0.3. Low dimensional Riemannian geometry is an extremely active area of mathematics. There are many open problems and applications. There are close relations to computer science and computer graphics. Taking the computer science point of view and sticking to coordinates is something, which many mathematicians scoff about. It is true that a coordinate free approach can feel more elegant, but it can also be just dream walking and talk about objects one has no intuition about. Especially for a beginner, it is good to be able to be able to see everything in a concrete manner at first. Going abstract afterwards is much easier.

0.4. Riemannian geometry is not only a prototype mathematical theory, it also helps to inspire other fields like topology or combinatorics. Topological statements like that a simply connected 3-manifold is topologically equivalent to a sphere was solved by adding a Riemannian metric structure on the manifold first. Combinatorial notions of manifolds have led to interesting combinatorial questions. We will see here how easy the discrete set-up is. A precise frame work for discrete manifolds including all differential geometry can be done in a few lines. Gauss-Bonnet for general networks is almost a joke when compared to the difficulties we encounter in the continuum.

0.5. From the application side, the subject is a marvel. The geometry of Riemannian manifolds is the lingua franca of gravity. Riemannian manifolds are also an inspiration for the arts. The actual metric implementation of a manifold is important in aesthetics. Both very smooth or polyhedral implementations can matter. When looking at the shape of objects like cars, houses, cloths or tools, one can see oscillations over time between smooth and edgy. Geometric considerations also matter more and more in data analysis. Artificial intelligence frame work embed knowledge in higher dimensional spaces and use distances to build large language models.

0.6. The subject is also saturated with unsolved problems. Simple sounding questions are unresolved like whether a positive curvature even dimensional manifold has positive Euler characteristic or whether there is a positive curvature metric on the product of two 2-spheres. We have even in the case of ellipsoids no answer yet about the number of cusps of caustics of wave fronts. How many closed geodesics are there for a given manifold? What are the Wiederkehr manifolds in higher dimensions, manifolds where all geodesics are closed. For which type of manifolds do spectral properties determine the manifold? Are two manifolds with isomorphic geodesic flows and equal diameter automatically isometric?

0.7. We had 20 lectures for this course. The goal was to reach four mountain peaks:

(1) the Frenet-Serret theorem
(2) the Gauss-Bonnet theorem
(3) the Theorema egregium
(4) the Einstein equations

We restrict to 2 pages per lecture because two pages are realistically teachable in an hour. The first three every first course in Riemannian geometry covers, the last has been reached sometimes here in the past. Unlike the first course in Riemannian geometry I was exposed when I was in college, we do not restrict to dimension 2. But Gauss-Bonnet is for 2-manifolds. The higher dimensional Gauss-Bonnet-Chern theorem is almost never proven, even in graduate courses.

0.8. Some literature consulted:

- W. Kühnel, Differential Geometry: Curves - Surfaces - Manifolds", 3. Edition.
- M. P. Do Carmo, Differential Geometry of Curves and Surfaces", 2. Edition. [1]
- M. Berger, "A panoramic view of Riemannian geometry". A marvel.
- S. Gudmundsson, "An introduction to Gaussian Geometry". (Nice 2023 notes).
- I.A. Taimanov, "Lectures on Differential geometry". (Refreshingly short).
- H.L. Cycon, R.G. Froese, W.Kirsch, B.Simon: "Schrödinger operators". (Ch.12).
- J.A. Thorpe, "Elementary Topics in Differential geometry".
- A. Pressley, "Elementary Differential geometry". (2. edition).
- M. Lipschultz, "Schaum Outline: differential Geometry".
- C.W. Misner, K.S. Thorne, and J.A. Wheeler "Gravitation".
- Y.Choquet-Bruhat: "General relativity and the Einstein equations".
- J.A. Wheeler: "Journey into gravity and space-time".
- Y. Choquet Bruhat and C. DeWitt-Morette: "Analysis, Manifolds and Physics".
- J. Marsden and T. Ratiu: "Manifolds, Tensor Analysis".
- O. Knill, "Introduction to geometry and geometric analysis". (1995 notes).

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[1] Both texts have been used traditionally in the last 20 years here at Harvard