DIFFERENTIAL GEOMETRY

MATH 136

Unit 3: Curves

3.1. **Curves** in $\mathbb{R}^n$ can be be either given as images of smooth maps $r : \mathbb{R} \to \mathbb{R}^n$ or as solutions $f = 0$ to $(n-1)$ equations $f : \mathbb{R}^n \to \mathbb{R}^{n-1}$. In the first homework, you have seen the intersection of $n-1 = 2$ surfaces $f_1 = 0$ and $f_2 = 0$ in $\mathbb{R}^3$ which gave the Viviani curve. Looking at **solution sets of equations** is more like a **algebraic geometry** thing. Here, in differential geometry, we primarily look at **parametrizations** $[a, b] \to \mathbb{R}^n$. An example of a curve is the **helix** $r(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ t \end{bmatrix}$ in $\mathbb{R}^3$. [1]

3.2. The **Jacobean matrix** of a curve $r(t)$ is $\boxed{dr(t)}$

$$r(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_n(t) \end{bmatrix}, dr(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \cdots \\ x_n'(t) \end{bmatrix} = r'(t) \; .$$

It is also known as the **velocity** and just abbreviated $\boxed{r'(t)}$. The **first fundamental form** is

$$g = dr^T dr = |r'(t)|^2 \; .$$

It is the square of the speed. The **arc length** of the curve is defined as

$$L = \int_a^b |r'(t)| \; dt \; .$$

Related to arc length is the **action**

$$I = \int_a^b |r'(t)|^2 \; dt \; .$$

which has the advantage that it can be computed better and produces equivalent variational problems. Minimizing the arc-length is equivalent to minimize the action and leads to geodesics. Here we are in flat Euclidean space and geodesics are straight lines. We will say more about this in class. You show in the homework:

Theorem 1 (Archimedes)**.** *The straight line is the shortest path connecting $A, B \in \mathbb{R}^n$.*

[1] We will often write also just $[\cos(t), \sin(t), t]^T$ or simply $[\cos(t), \sin(t), t]$ without the transpose for typographic reasons.

3.3. A curve is called **simple** if r is it does not have self intersections. It is called **regular** if the first fundamental form is nowhere zero. Equivalently, this means that the velocity is nowhere zero. A simple closed curve in space is called a **knot**. An example is the **figure 8 knot**

$$r(t) = [(2+\cos(2t))\cos(3t), (2+\cos(2t))\sin(3t), \sin(4t)]^T$$

parametrized on $[0, 2\pi]$. We talk more about this in class like that it lives on a torus and why you can not tie knots in $\mathbb{R}^n$ for $n > 3$. r is **simple** can be rephrased that the map $r : [0, 2\pi) \to \mathbb{R}^3$ is **injective**.

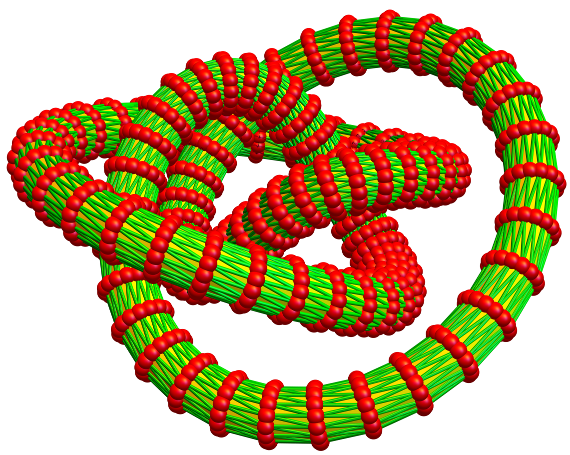

FIGURE 1. A picture of a knot. Drawing the curve in such a fancy way, needs concepts like curvature and torsion, which we will learn next week. The first fundamental form of the figure 8 knot is $r'(t)^2 = 101/2 + 36\cos(2t) + (5/2)\cos(4t) + 8\cos(8t)$. The action is $I = 101\pi$, the arc length involves elliptic integrals. Numerically it is $L = 42.966\ldots$. It is typical that we can explicitly give the action but not the length.

3.4. A curve is **parametrized by arc length** if $|\vec{r}\,'(t)| = 1$ for all t. You will prove in homework the following important result:

Theorem 2. *Every smooth regular curve in* $\mathbb{R}^n$ *can be parametrized by arc-length.*

3.5. It is custom to write $r(s)$ to indicate that we have an arc length parametrization. For the helix above, the arc length parametrization is $r(s) = [\cos(s/\sqrt{2}), \sin(s/\sqrt{2}, 1/\sqrt{2}]$. In general we do not bother to actually compute the arc length parametrization. Already in simple cases like the ellipse it would get nasty. We can use the theorem however to build theory and prove stuff about curves.

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