DIFFERENTIAL GEOMETRY

MATH 136

Unit 6: Hopf Umlaufsatz

6.1. We look today at a theorem in two dimensions. It deals with **signed curvature** $\kappa = \frac{r' \times r''}{|r'|^3}$ using the cross product in 2 dimensions. [1]. We do not assume that the curve is Frenet. The curvature is allowed to become zero. We assume however that the curve is closed and regular meaning that $dr = r'$ is never zero. In that case, there is an arc length parametrization of the curve and $|\kappa| = |r''|$ because $r' \cdot r' = 1$ implies r'' is perpendicular to r'. But we have a signed curvature!

6.2. Assume that the curve is parametrized on $[a, b]$. The **rotation index** is defined as $\frac{1}{2\pi}\int_a^b \kappa(t)\, dt$. If the closed curve is not arc length parametrized, this is $\int_a^b \kappa(t)|r'(t)|\, dt$.

Theorem 1. *The rotation index of a closed C^2 curve is in $\mathbb{Z}$.*

Proof. Using arc length parametrization, write

$$r'(t) = [\cos(\alpha(t)), \sin(\alpha(t)]$$

then $\kappa = \alpha'$. Since the curve is closed, we have $\alpha(b) - \alpha(a) = 2\pi n$, where n is an integer. □

6.3. The case $r(t) = [\cos(nt), \sin(nt)]$ with $t \in [0, 2\pi]$ shows that the rotation index can take any integer value n. It is intuitively clear that if a curve has no self intersections, then the index must be either 1 or -1. This is not so obvious however. We do not want for example to refer to the Jordan curve theorem telling that a continuous simple closed curve in the plane divides the plane into an inside and outside. Heinz Hopf found a nice argument which proves this "Umlaufsatz" in an elegant way using a deformation picture:

Theorem 2 (Hopf Umlaufsatz)**.** *A simple closed regular C^2 curve has rotation index 1 or -1.*

Proof. Arc length parametrization is not needed. We assume that $r(t)$ is parametrized on the interval $[0, 1]$. Define on the square $Q = [0, 1] \times [0, 1]$ the function $f : Q \to \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})\}$ given by $f(t, s) = \arg((r(t) - r(s))/|r(t) - r(s)|)$ for $t \neq s$ and $\alpha(t) = f(t, t) = \arg(r'(t)/|r'(t)|)$ for $t = s$. Because $r \in C^1$, the function f is continuous. Now comes a homotopy argument. The index is $[f(1, 1) - f(0, 0)]/(2\pi)$ and is an integer. If we move along the diagonal and look at $\alpha(t) = f(t, t)$ we see a continuous curve which

[1]The cross product in n dimensions has $\binom{n}{2} = n(n-1)/2$ components. For $n = 2$ it is a scalar

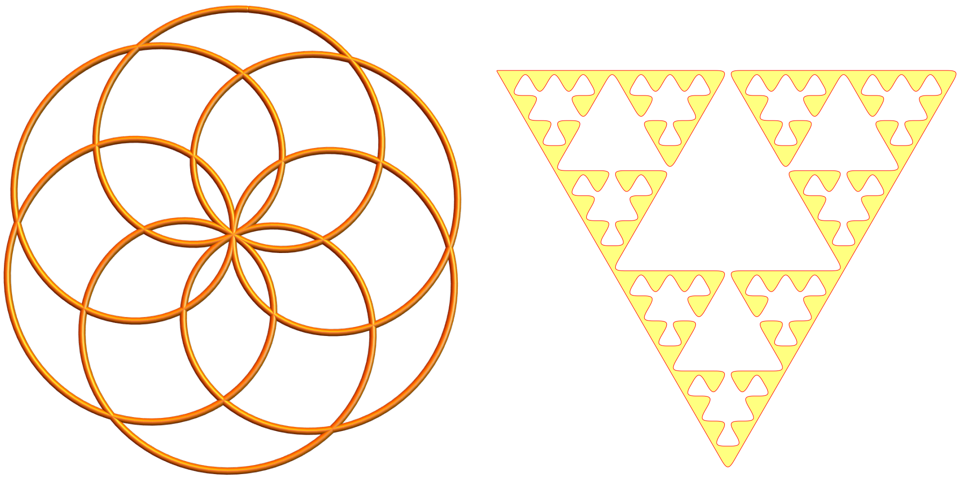

FIGURE 1. To the left the curve $r(t) = [\cos(t) + \cos(7t), \sin(t) + \sin(7t)], t \in [0, 2\pi]$ reminds of the flower of life. Its rotation number is 7. We can compute $\kappa(t)|r'(t)| = 4 + 72/(25 + 8\cos(6t))$ which integrates on $[0, 2\pi]$ up to 14π. To the right, a simple closed smooth curve in the plane. What is its rotation number?

moves on the circle $\mathbb{T}$. If we deform the curve the total change remains the same. We can continuously deform the curve so that we first deform from $(0,0)$ straight to $(0,1)$ and then straight from $(0,1)$ to $(1,1)$. Choose a coordinate system so that is in $y \geq 0$ just touching the x-axes. If $r'(0) = [a, 0]$ with positive a then $f(t,s) \in [0,\pi]$ with $f(0,0) = 0$ and $f(0,1) = \pi$ and then $f(1,1) = 2\pi$. If $a < 0$, then $f(t,s) \in [-\pi, 0]$ with $f(0,0) = \pi$ and $f(0,1) = 0$ and then $f(1,1) = -\pi$. In the former case, $i = 1$ in the later $i = -1$. $\square$

6.4. Remarks:
1) This is a Gauss-Bonnet type result for a 2 dimensional flat manifold with boundary.
2) The proof shows that this even works for C^1 curves as $f(t,t) - f(s,s)$ is just the angle change of the tangent. This works even if the curvature is not defined. In the homework you even push it to polygons. Most texts assume C^2.

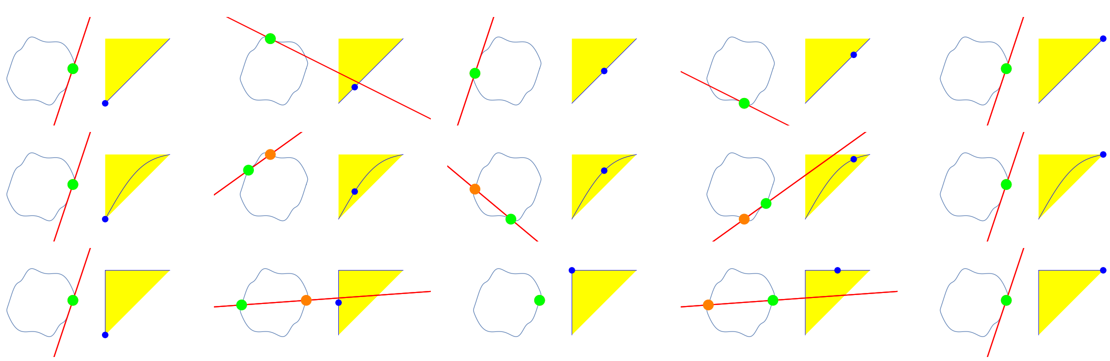

FIGURE 2. The deformation argument.

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