DIFFERENTIAL GEOMETRY

MATH 136

Unit 7: Four vertex theorem

7.1. A **vertex** is a local maximum or minimum of the curvature function $\kappa(t)$. If you look at the case of an ellipse that is not a circle, you see two maxima and two minima. There are therefore 4 vertices. What happens in general? A curve is called **convex** if it bounds a convex region R. A region R is called **convex** if the line segment between any two points $A, B \in R$ is part of the region.

Theorem 1. *A simple closed regular convex C^3 plane curve has at least 4 vertices.*

Proof. We can assume κ is not constant. As a continuous function on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ it by the extremal value theorem at least one maximum a and one minimum b. Assume, $r(a)$ and $r(b)$ were the only critical points and that $[0, L]$ is the parameter interval. Choose the coordinate system so that these two points $r(a), r(b)$ are on the x-axis so that $r(t) = [x(t), y(t)]$ and $y(a) = y(b) = 0$. Convexity assures that there is no other root $y(t) = 0$. So, k' changes sign at $s = a$ and $s = b$ and nowhere else and $k'(s)y(s)$ does not change sign at all. The Frenet equations tell $e_1 = T = [x', y'], e_2 = N = [-y', x'], [x'', y''] = e_1' = \kappa e_2 = \kappa[-y', x']$ from which follows that $x'' = -\kappa y'$. Integration by parts gives $\int_a^{a+L} \kappa'(s)y(s)\, ds = \kappa y|_0^L - \int_a^{a+L} \kappa(s)y'(s)\, ds = \int_a^{a+L} x''(s)\, ds = x'(a+L) - x'(a) = 0$ which is not possible given that $\kappa'(s)y(s)$ does not change sign. There is therefore an other maximum or minimum. The number of local max and local min are the same for a periodic function (they must alternate as two successive maxima have a minimum between) so that there must be 4 vertices. □

7.2. Examples.
1) The ellipse $r(t) = [2\cos(t), \sin(t)]$ has the curvature $\kappa(t) = 2(\cos^2(t) + 4\sin^2(t))^{-3/2}$ which has maxima at $t = 0, \pi$ and minima at $t = \pm\pi/2$.
2) The curve $r(t) = 5[\cos(t), \sin(t)] + [\cos(2t), \sin(2t)]$ has curvature $r'(t) \times r''(t)/|r'(t)|^{3/2}$ that has minima at $0, \pi$ and maxima at $\pm \arccos(-2/5)$.
3) The **limacon** $r(t) = [\cos(t), \sin(t)] + [\cos(2t), \sin(2t)]$ has curvature with only 2 vertices! Why is this not a counter example?

7.3. Remarks.
1) Convexity is not really needed. Osserman showed that for any simple closed C^2 curve C there are $2n$ components if the smallest circle enclosing C intersects it in at least n connected components. A general simple closed C^2 curve has at least 4 vertices.
2) V. Arnold conjectured that the result holds for any curve that can be obtained from a circle by suitable deformations.

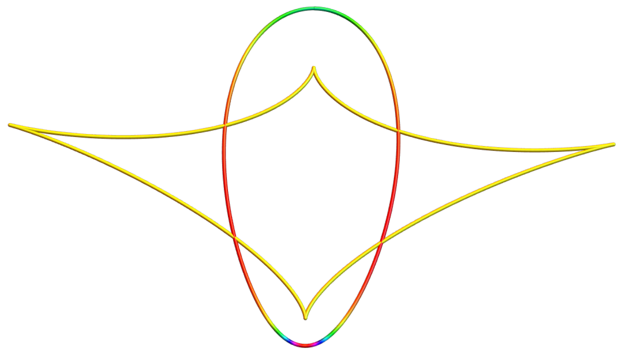

Figure 1. A simple closed curve. Color encodes curvature. The 4 vertices are visualized by plotting the **evolute** $e(t) = r(t) + n(t)/\kappa(t)$, where $n(t)$ is the normal vector pointing inside. The vertices of the curve correspond to cusps of the evolute.

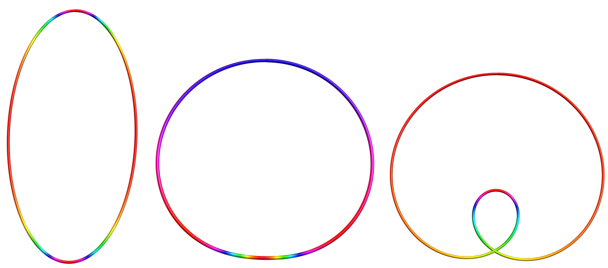

Figure 2. The ellipse, a deformation of an ellipse and a limacon.

3) If a continuous real valued, periodic function has at least two local maxima and two local minima, it is the curvature function of a simple closed curve.

7.4. History.
1) The theorem was proven in 1909 by Syamadas Mukhopadhyaya for convex curves. 2) The general case was published in 1912 by Adolph Kneser. 3) The proof given above in the convex case is due to G. Herglotz in 1930. 4) Robert Osserman in 1985 generalized the result to the "four or more vertex theorem". 5) The converse result started with H. Gluck in 1971 and was proven in 1997 by Bjoern Dahlberg.

7.5. Related.
1) The **evolute** of a plane curve is defined as $e(t) = r(t) + n(t)/\kappa(t)$. It is the caustic of the normal map. At points where $\kappa'(t)$ is zero, the evolute has cusps. A caustic of a smple closed curve therefore has at least 4 cusps. For the ellipse the evolute is called the Lamé curve.
2) The **tennis ball theorem** states that a C^2 curve on the sphere that divides the sphere into regions of equal area have at least 4 inflection points.
3) The open **last geometrical problem of Jacobi** asks whether a caustic on an ellipse has at least 4 cusps.

Oliver Knill, knill@math.harvard.edu, Math 136, Fall, 2024