Unit 12: The exponential map

12.1. The geodesic differential equation $\boxed{\ddot{x}^k + \sum_{i,j} \Gamma^k_{ij} \dot{x}^i \dot{x}^j = 0}$ can be written as a first order system $\frac{d}{dt}[x, \dot{x}] = [\dot{x}, f(x, \dot{x})]$ if the first fundamental form g is twice differentiable. This ordinary differential equation therefore has **local solutions** for some time $t \in (-a, a)$ by the Piccard existence theorem. But the solutions exist for all time. No "blow up" is possible if the surface is smooth, regular and closed. The reason is the following:

Lemma 1. *If $x(t)$ is geodesic, then $\langle \dot{x}, \dot{x} \rangle = \sum_{i,j} g_{ij} \dot{x}^i \dot{x}^j$ is preserved.*

Proof. Either note $\frac{d}{dt}\langle \dot{x}, \dot{x} \rangle = 2\langle \ddot{x}, \dot{x} \rangle = 0$. Alternatively, define the Hamiltonian $H = -F + \sum_j \dot{x}^j F_{\dot{x}^j}$. Using the Euler-Lagrange equations, we get $\frac{d}{dt}H = -\sum_j F_{x^j} \dot{x}^j - \sum_j F_{\dot{x}^j} \ddot{x}^j + \sum_j \ddot{x}^j F_{\dot{x}^j} + \sum_j \dot{x}^j \frac{d}{dt} F_{\dot{x}^j} = 0$. (Now replace the last term with $\sum_j \dot{x}^j F_{x^j}$.) For $F(x, \dot{x}) = \langle \dot{x}, \dot{x} \rangle$, we have $H = -F + 2\sum_j \dot{x}^j \dot{x}^j = -F + 2F = F$. We see $H = F = \langle \dot{x}, \dot{x} \rangle$ is an energy that is preserved. □

Theorem 1 (Hopf-Rynov)**.** *For regular, compact, smooth M, geodesics exist globally.*

Proof. If M is C^4 then Γ is differentiable. The Piccard existence theorem gives local solutions in the unit tangent bundle $(p, v) \in M \times S^{m-1}$. A regular compact manifold is complete in the sense that all Cauchy sequences have limits. The only way that a solution path could not be continued is that $\dot{x}(t)$ blows up. Otherwise, we could restart the differential equation at the end point a of a maximal interval $(-a, a)$ of existence. By the lemma, a blow up of $\dot{x}(t)$ is not possible. □

12.2. Remarks: **a)** The regularity is necessary. On a piece-wise smooth manifold like a cube, a geodesic hitting a corner can not be continued continuously. **b)** There are compact Lorentzian manifolds like the Clifton-Pohl torus that are not complete. **c)** The lemma is important. The proof shows that each variational problem gets with a **Legendre transform** $H = -F + \sum_j \dot{x}^j F_{x^j}$ to an "energy" H that is preserved. [1]

12.3. The **exponential map** $\exp_p : T_pM \to M$ is obtained by defining $\exp_p(0) = p$ and for $v \neq 0$, define $\exp_p(v)$ by taking $v/|v|$ as initial direction of the geodesic flow and evolving it for time $|v|$. The image $\exp(S_r(0)) = W_r(p)$ is called the **wave front**. It is the set of all points which can be reached from p by running from it a geodesic of length p. Wave fronts are **geodesic circles** for small t but in general become very complicated.

[1]For more, see J. Moser, Selected Topics in the Calculus of Variations. (Notes by O. Knill) 2002

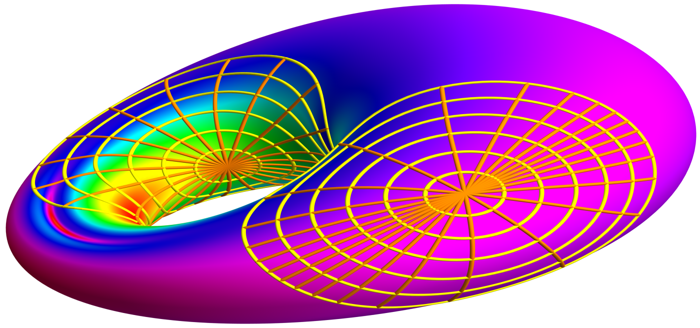

FIGURE 1. The exponential map evolves all possible geodesics from p. If all these geodesics are stopped at time t, we get a wave front $W_t(p)$.

12.4. The **radius of injectivity** of M is the smallest r such that the exponential map $B_r \subset T_pM \to M$ is injective.

Lemma 2. *For a compact manifold M, the radius of injectivity is positive.*

Proof. When fixing a point p, there is a $B_r(0) \subset \mathbb{R}^m$ such that that $\exp_p$ is invertible. This follows from the **inverse function theorem** and the fact that $d\exp_p = 1$ (identity matrix) at p because $\exp_p(v) - v = O(|v|^2)$ by definition. Let $r(p)$ be maximal radius on which $\exp_p(B_r(0))$ is differentiable. This function $r(p)$ is continuous in p and positive. By compactness of M and the **extremal value theorem**, there is a minimum, a lower bound. □

12.5. For fixed p, critical values of $\exp_p$ form the **caustic** of p. If r is the radius of injectivity, the open set $U = \exp_p(B_r(0)) \subset M$ is called the **normal neighborhood** of p. Lets look at the two dimensional case:

Lemma 3. *On U there are coordinates (ρ, θ) such that $g = I = \begin{bmatrix} 1 & 0 \\ 0 & G \end{bmatrix}$ satisfying $\lim_{\rho \to 0} G(\rho, \theta) = 1$.*

Proof. These are called **geodesic polar coordinates** because they come from the exponential map. Since velocity is preserved, the radial direction does not expand. □

12.6. This implies:

Theorem 2 (Gauss Lemma)**.** *For every unit vector v, the radial geodesics $\{\exp_p(sv), s \leq t\}$ is normal to the wave front $W_t(p)$.*

Proof. Within $U = \exp_p(B_r(0))$ this is clear by the coordinates. □

12.7. Remarks. 1) Geodesic coordinates with $I = g = \text{diag}(1, g_{22} \dots, g_{mm})$ exist on any m-manifolds. **2)** For 2-manifolds, linearising the geodesic flow affects only the vector perpendicular to the geodesic $x(t)$. This is called a **Jacobi field**. For surfaces, for fixed p and v, we get a **Jacobi differential equation** $\boxed{z'' = -K(x(t))z}$, where $z(t) = G(x(t))$ in the normal patch. The roots of $z(t)$ belong to caustic points $\exp_t(p)$.