DIFFERENTIAL GEOMETRY

MATH 136

Unit 13: Curvature is a Curl

13.1. The proof of the **Gauss-Bonnet theorem** will invoke **Green's theorem** from calculus. Also the **Theorema egregium** will boil down to the fact that curvature form KdV is the curl dX of a 1-form X, that only depends on the first fundamental form I. **Differential geometry** so builds heavily on **multi-variable calculus**.

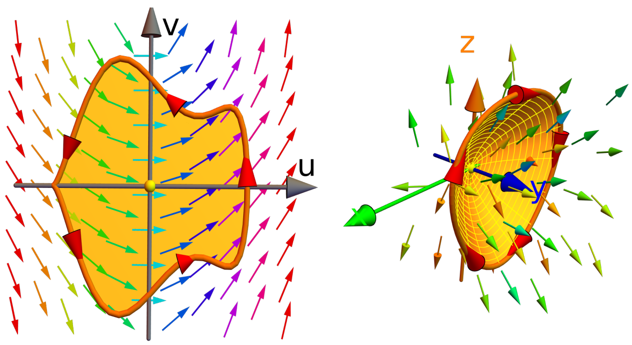

FIGURE 1. **Green's theorem** integrates the 2D curl dX over a planar region R. **Stokes theorem** integrates the 3D curl dF over a surface M. If $M = r(R)$, one can **pull back** the 1-form F in $\mathbb{R}^3$ to a 1-form X in $\mathbb{R}^2$ and so get Stokes from Green: $dF(r_u, r_v) = \text{curl}(F) \cdot r_u \times r_v = F_u \cdot r_v - F_v \cdot r_u = \text{curl}(X)$ for $X = [F \cdot r_u, F \cdot r_v]$ (see homework). In differential geometry, a particular X will lead to Gauss-Bonnet.

13.2. Green's theorem is usually written for planar vector fields $X^T = \begin{bmatrix} P \\ Q \end{bmatrix}$: the double integral of the curl dX of X in a R agrees with the line integral of X along the boundary δR. If we change to row vectors, we have a **1-form** $X = [P, Q]$. 'Power=force times velocity" $\begin{bmatrix} P \\ Q \end{bmatrix} \cdot \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix}$ is then the **matrix product** $X\dot{x}$ rather than $X^T \cdot \dot{x}$. [1]

13.3. Assume $X = [P, Q]$ is a 1-form and assume $x(t) = \begin{bmatrix} u(t), v(t) \end{bmatrix}^T$ is the parameterization of a **closed curve** δU with $t \in [0, L]$ bounding the region $U \subset R$. The **curl** of X is defined as $dX = \text{curl}(X) = Q_u - P_v$. The **1-form** X is a linear map which assigns to a contra-variant vector (column vector) like $\dot{x}$ a number $X\dot{x} = P\dot{u} + Q\dot{v}$.

[1] Both in physics as well in differential geometry, it is important to distinguish between **contra-variant objects** like vector fields $\nabla f = X^i$ and **co-variant objects** like 1-forms $df = X_i = \partial_{x^i} f$.

A 1-form especially can be integrated along a curve $\int_0^L X(x(t))\dot{x}dt$, the **line integral**. The curl of X is denoted by dX. It is a **2-form** which can be integrated over U. The **general Stokes theorem** tells $\boxed{\int_M dX = \int_{\delta M} X}$ if M is a k-manifold with boundary δM and X is a $(k-1)$-form then dX is a k-form. In the case $k=2$, where X is a 1-form and $U \subset R$ is a region, we have

Theorem 1 (Green)**.** $\int_{\delta U} X \; = \int_0^{2\pi} X(x(t))\dot{x}(t)\; dt = \iint_U \text{curl}(X)(u,v)\; dudv = \int_U dX.$

13.4. In calculus, you see this using vector fields $F = X^T$, meaning that every point is attached a contra-variant vector. In order to pair this with the velocity vector $\dot{x}$, we had to invoke the **dot product** $v \cdot w = v^T w$ and write a **matrix product** $X(x(t))\dot{x}(t)$. The just formulated version of Green's theorem is completely equivalent.

13.5. The key of Gauss Bonnet is to see that the curvature 2-form $K|r_u \times r_v|$ can be written as the curl dX of a 1-form X. Gauss-Bonnet theorem in the convex case is stated as $\iint_R K|r_u \times r_v|duv = 2\chi(M)$. A second computation will then show that if $M = r(U)$ is a manifold with boundary $r(x) = \delta(M)$, integrating the geodesic curvature along the boundary curve x is a **line integral** of X along x plus 2π. Gauss-Bonnet for surface patches $r(U)$ with boundary $t \to r(x(t))$ will then follow from Green's theorem.

13.6. Assume that $r : R \to \mathbb{R}^3$ is a regular parametrization of the surface M. A simple closed curve $x(t), t \in [0,L]$ encloses a region $U \subset R$ matching orientation. It defines a curve $r(x(t))$ bounding the manifold $r(U) \subset M$. We can assume that $x(t) = (u(t), v(t))$ is parametrized by arc length. At every point $p = r(u,v) \in M$, the vectors $\{r_u, r_v\}$ form a basis of the tangent space T_pM. Let $\{z, w\}$ be the Gram-Schmidt orthonormalized basis obtained from $\{r_u, r_v\}$ and the unit normal vector $n = r_u \times r_v/\sqrt{r_u \times r_v} = z \times w$.

13.7. The following lemma shows that we can attach two vectors z, w to every point p on the surface. It will allow us to define the 1-form $\boxed{X = zdw = [z \cdot w_u, z \cdot w_v]}$.

Lemma 1. *$z = ar_u, w = br_u + cr_v, n = z \times w$ form an orthonormal frame with functions a, b, c that only depend on the first fundamental form.*

Proof. Gram-Schmidt proceeds as follows $z = r_u/\sqrt{r_u \cdot r_u} = r_u/\sqrt{E} = ar_u$ and gets w as the normalization $br_u + cr_v$ of $r_v - (r_v \cdot z)z = r_v - (r_v \cdot r_u)r_u/E = r_v - \frac{F}{E}r_u$. □

13.8. We will see next time that X can be computed from I alone and that

Lemma 2 (Curvature is a curl)**.** *The curl satisfies* $\boxed{dX = Q_u - P_v = K\sqrt{\det(g)}}$.

13.9. For now, this is just an announcement. The computation comes next class. But then we will be close to Gauss-Bonnet: the line integral of X along the boundary will then be related with an integral of geodesic curvature so that we will reach the local Gauss-Bonnet theorem $\int_M X = \int_M K\; dV = \int_C dt - \kappa_g ds = 2\pi - \int_C dX$. And then by gluing, we will get the **global Gauss-Bonnet theorem** $\int_M X = 2\pi\chi(M)$. This is the mountain peak we wanted to reach. We are in the middle of the climb right now.

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