DIFFERENTIAL GEOMETRY

MATH 136

Unit 18: Discrete Manifolds

18.1. A **discrete m-manifold** is a finite graph $G = (V, E)$ for which every unit sphere $S(v)$ is a discrete $(m-1)$-sphere. A **discrete m-sphere** is a discrete m-manifold which has the property that removing a point renders it contractible. Inductively, a graph is called **contractible**, if both $S(v)$ and $S \setminus v$ are contractible for some $v \in V$. The 1-point space 1 is contractible. The empty graph is the (-1)-sphere. Let F_k denote the set of K_{k+1} subgraphs (k-simplices) and $f_k = |F_k|$. We have $F_0 = V, F_1 = E$. The **Euler characteristic** of M is defined as $\chi(M) = \sum_{k=0}^{m}(-1)^k f_k = f_0 - f_1 + f_2 - f_3 + \cdots + (-1)^m f_m$. This definition of Ludwig Schläfli generalizes $\chi(M) = f_0 - f_1 + f_2 = V - E + F$ for 2-manifolds.

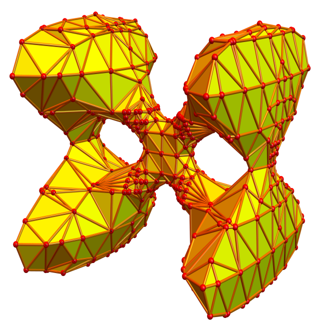

FIGURE 1. This 2-manifold M of genus $g = 2$ has $\chi(M) = 2 - 2g = -2$.

18.2. A graph without edges is a 0-manifold. A 0-manifold is a 0-sphere, if $V = 2, E = 0$ (removing a vertex produces K_1 which is contractible by definition). Every connected 1-manifold is a 1-sphere, a circular graph C_n with $n \geq 4$. Every finite 2-manifold is either a 2-sphere S^2 or a connected sum of tori or projective planes: $M = S^2, M = \mathbb{T}^2 \# \cdots \# \mathbb{T}^2$ or $M = \mathbb{P}^2 \# \cdots \# \mathbb{P}^2$. A 2-sphere can be characterized as 2-manifold of Euler characteristic 2. The 16 cell and the 600 cells are examples of 3-spheres. The join of two 1-spheres is a 3-sphere. The join of a k-sphere with a m-sphere is a (k+m+1)-sphere. The join of G with the 0-sphere is called **suspension**.

18.3. Euler's formula $\chi(M) = V - E + F = 2$ for 2-spheres generalizes to higher dimension. The 0-sphere has $\chi(M) = V = 2$, every 1-sphere has $\chi(M) = V - E = 0$. Every 2-sphere has $\chi(M) = V - E + F = 2$. This pattern continues:

Theorem 1 (Euler's Gem)**.** *If M is a m-sphere, then $\chi(M) = 1 + (-1)^m$.*

Proof. Use induction with respect to dimension m. For $m = 0$, we have $\chi(M) = 2$. The induction assumption is that all $(m-1)$-spheres S satisfy $\chi(S) = 1 + (-1)^{m-1}$. Pick a vertex v. As the unit sphere $S(v)$ is a $(m-1)$-sphere and $S(v) = B(v) \cap G \setminus v$, where both the unit ball $B(v)$ and $G \setminus v$ are contractible with Euler characteristic 1, we have, using the induction assumption, $\chi(M) = \chi(G \setminus v) + \chi(B(v)) - \chi(G \setminus v \cap B(v)) = 2 - (1 - (-1)^{m-1}) = 1 + (-1)^m$. □

18.4. In the continuum, manifolds can be constructed as level surfaces of functions like $x^2 + y^2 + z^2 = 1$. We can do that also in the discrete. Take an arbitrary function on vertices V which takes values in $Z_k = \{0, \dots, k\}$. It defines a new graph M_f, where the vertices are the set of complete subgraphs on which f attains all k values. Connect two of these points by an edge, if one is contained in the other. The new graph M_f is a sub-graph of the **Barycentric refinement** of M. Here is the analog of what we have seen classically for functions on manifolds. It surprises that singularities like in the Viviani curve (HW 1) do never occur in the discrete:

Theorem 2 (Level Sets)**.** *If M is a m-manifold and $f : M \to Z_k$ is an arbitrary function, then either M_f is empty or then M_f is a $(m-k)$-manifold.*

Proof. Let x be a n-simplex on which f takes all values. This means $f(x) = Z_k$. The graph $S^-(x) = \{y \subset x, y \neq x\}$ is a $(n-1)$-sphere in the Barycentric refinement of M. The simplices in $S^-(x)$ on which f still reaches Z_k is by induction a $(n-1-k)$-manifold and since we are in a simplex, it has to be a $(n-1-k)$-sphere. Every unit sphere $S(x)$ in the Barycentric refinement is a $(m-1)$-sphere as it is the join of $S^-(x)$ with $S^+(x) = \{y, x \subset y, x \neq y\}$. (The join of two spheres is always a sphere.) The sphere $S^+_f(x)$ in M_f is the same than $S^+(x)$ in M because every simplex z in M containing x automatically has the property that $f(z) = Z_k$. So, the unit sphere $S(x)$ in M_f is the join of a $(n-k-1)$-sphere and the $(m-n-1)$-sphere and so a $(m-k-1)$-sphere. Having shown that every unit sphere in M_f is a $(m-k-1)$-sphere, we see that M_f is a $(m-k)$-manifold. □

18.5. What about differential geometry? No problem. Define **curvature** as

$$K(v) = \sum_{k=0}^{m} \frac{(-1)^k f_{k-1}(S(v))}{k+1} = 1 - \frac{f_0(S(v))}{2} - \frac{f_1(S(v)))}{3} + \dots .$$

In the case of a 2-manifold this boils down to $1 - f_0(S(v))/2 + f_1(S(v))/3 = 1 - d(v)/6$, where $d(v)$ is the vertex degree. For odd-dimensional manifolds, the curvature is constant zero. You experiment with this in Homework 10.

Theorem 3 (General Gauss-Bonnet)**.** $\sum_{v \in V} K(v) = \chi(M)$

Proof. The proof is the same as in the 2-dimensional case. Again look at the energies $\omega(x) = (-1)^{\dim(x)}$ attached to each simplex x in the graph (complete subgraph with $\dim(x) + 1$ vertices). Then $\chi(M) = \sum_x \omega(x)$. Now distribute all these energies of a k-simplex x equally to the $k+1$ vertices contained in x. As there are f_{k-1} simplices in $S(v)$ which correspond do simplices containing v, this adds $\frac{(-1)^k f_{k-1}(S(v))}{k+1}$ to each vertex v. Now just collect all up at a vertex v to get the curvature $K(v)$. The transactions of energies preserved the total energy = Euler characteristic. □