Notations

ALGEBRA

99.1. $\mathbb{R}^n$ is the **Euclidean space** in n dimensions. A finite rectangular array A of real numbers is called a **matrix**. If there are n rows and m columns in A, it is called a $n \times m$ matrix. The set of all theses matrices $\mathbb{R}^{n,m}$ forms a vector space. The entry of A_i^j is in the i'th row and j'th column. A $n \times 1$ matrix is a **column vector = (1,0) tensor**, a $1 \times n$ matrix is a **row vector= (0,1) tensor**. A 1×1 matrix is a **scalar = (0,0) tensor**. For a $n \times p$ matrix A and a $p \times m$ matrix B, the $n \times m$ **matrix product** AB is defined as $(AB)_i^j = \sum_{k=1}^p A_i^k B_k^j$. The **transpose** of a $n \times m$ matrix A is the $m \times n$ matrix $(A^T)_i^j = A_j^i$. The transpose of a column vector is a row vector. There are various types of matrices. Square matrices are the case $n = m$ ((1,1)-tensor), symmetric matrices satisfy $A_i^j = A_j^i$ for all i, j. There is an **inner product** $A \cdot B = \text{tr}(A^T B) = \sum_{i,j} A_i^j B_j^k$ on the space of matrices. With an inner product, we have a length $|A| = \sqrt{A \cdot A}$. The Cauchy-Schwarz inequality is $|A \cdot B| \leq |A||B|$ and allows to define angles $\cos(\alpha) = A \cdot B/(|A||B|)$ between two nonzero matrices of the same type. This generalizes the usual **dot product** for vectors.

TOPOLOGY

99.2. A **topological space** is a set X on which one has a set of subsets $\mathcal{O}$ called **open sets** in X. One assumes that $\emptyset, X \in \mathcal{O}$, that arbitrary unions from $\mathcal{O}$ are in $\mathcal{O}$ and that finite unions of elements in $\mathcal{O}$ are in $\mathcal{O}$. The complement of an open set is called closed. Sets that are both open and closed are **clopen**. The empty set and X are always clopen. If $x \in U$, then U is also called a **neighborhood** of x. The topological space is called **Hausdorff**, if $x \neq y$ are points in M, then there exist $U, V \in \mathcal{O}$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. A **metric space** (X, d) defines a topological space. Start with the open balls $B_r(x) = \{y, d(x, y) < r\}$ and assume them to be in $\mathcal{O}$. Then take the smallest topological space which contains all these open sets. This is the topology generated by the metric. A metric space is always Hausdorff because $x \neq y$ with $r = d(x, y) > 0$ implies that $B_{r/3}(x) \cap B_{r/3}(y) = \emptyset$.

99.3. Euclidean spaces like $\mathbb{R}^n$ or $\mathbb{R}^{n,m}$ have a natural distance $d(x, y)$ on them, given as the length of the vector from x to y, if the length or magnitude of a vector is defined as $\sqrt{v \cdot v}$. For a matrix this is defined by $d(A, B)^2 = \sum_{i,j} (A_{ij} - B_{ij}^2)$. There are other distance functions like if the dot product is allowed to change from point to point. Euclidean space equipped with a distance becomes a metric space and so have

a **topology**, meaning that we can form open sets, which are also called **neighborhoods**. In a metric space the open balls $B_r(P) = \{Q, d(P, Q) < r\}$ are open sets and form a basis for all open sets. Given a topology, we also have a notion of continuity meaning that the inverse of an open set is open. For metric spaces, this is equivalent to $\lim_{x_n \to x} f(x_n) = f(x)$ if a sequence of points x_n converges to x. A subset G of Euclidean space is called open if every point x has an open ball $B_r(x) = \{y, d(x, y) \leq r > 0\}$ which is contained in G. The complement of an open set is called **closed**. A subset of an Euclidean space is called **connected** if it can not be written as a disjoint union of two open sets. In other words, if $\emptyset$ and X are the only clopen sets, then X is connected.

CALCULUS

99.4. If f is a differentiable map from $\mathbb{R}^m$ to $\mathbb{R}^n$, its derivative df at x is the matrix $df(x) \in R^{n,m}$, the **Jacobean matrix** at x. Its entry is $df_{ij} = \partial f_i / \partial x_j$. For example, if $r : \mathbb{R} \to \mathbb{R}^n$ defines a curve, then r' is a column vector giving the velocity of r. In the case $n = 1$, this is the derivative in calculus. If $r : \mathbb{R}^> \to \mathbb{R}$ is a function of p variables, then $df = [\frac{\partial}{\partial x_1} f, \ldots, \frac{\partial}{\partial x_p}]$ is the **gradient**. If $f : \mathbb{R}^p \to \mathbb{R}^n$ is another function with $df(y) \in \mathbb{R}n, p$, we can combine them and form $f \circ r(x) = f(r(x)) : \mathbb{R}^m \to \mathbb{R}^n$. The matrices $df(y) \in \mathbb{R}^{n,p}$ and $dr(x) \in \mathbb{R}^{p,m}$ combine to the matrix product $df \, dr$ at a point, a matrix in $\mathbb{R}^{n,m}$. The **multi-variable chain rule** is: $d(f \circ r)(x) = df(r(x))dr(x)$. As df can again be seen as a map between Euclidean spaces, we can differentiate again if this is possible. If we can differentiate k times and get a continuous function, we say $f \in C^k(\mathbb{R}^m, \mathbb{R}^n)$. In differential geometry, we usually assume that functions are **smooth**, meaning that we can differentiate as many times as we want. If there is a differentiable map $f : M \to N$ for which there is a differentiable inverse map g, then f is called a **diffeomorphism**.

ANALYSIS

99.5. The **implicit function theorem** tells that if $f(x, y)$ is a function of two variables and $\partial_y f$ is invertible at x_0, then $y = g(x)$ near x_0. The **inverse function theorem** tells that if $f : \mathbb{R}^n \to \mathbb{R}^n$ has an invertible Jacobean matrix at x_0, then f is invertible near x_0. It follows that if $f : \mathbb{R}^m \to \mathbb{R}$ is a map such that df is nowhere zero on all points $\{f = 0\}$, then $\{f = 0\}$ is a $(m-1)$-dimensional submanifold. More generally, if $f : \mathbb{R}^n \to \mathbb{R}^k$ has the property that df has maximal rank k at every point of $\{f = c\}$ then $\{f = c\}$ is a $(n - k)$-dimensional sub-manifold. The space $\mathbb{R}^n \times \mathbb{R}^n$ is called the **tangent bundle** of $\mathbb{R}^n$. If M is a k-dimensional submanifold given by $\{f = c\}$ then the tangent space $T_p M$ at a point consists of all points perpendicular to the normal vector df^T. It is the kernel of $\ker(df)$. The normal space $\perp_p M$ is spanned by the vector df^T. For example, if $f(x, y, z) = x^2 + y^2 + z^2$, where $M = \{f = 3\}$ is a sphere and $df = [2x, 2y, 2z]$ is the gradient, the transpose $df^T = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$ is normal the manifold. The tangent space at a point like $P = (1, 1, 1)$ is the set of vectors perpendicular to the normal vector $[2, 2, 2]^T$.