

DIFFERENTIAL GEOMETRY

MATH 136

Midterm Assignment Due Friday 10/18/2024

Handwritten. No internet, no correspondence, no computer algebra system, no AI. Put your name on each page of your paper and acknowledge: **"I affirm my awareness of the standards of the Harvard College Honor Code."**

Problem A (20 points): The curve

$$r(t) = \begin{bmatrix} t \\ \cosh(t) \\ 0 \end{bmatrix}$$

is parametrized on the interval $[a, b] = [-1, 1]$. It is called the "catenoid" or "chain curve". You can use that the function $\cosh(t) = (e^t + e^{-t})/2$ has the derivative $\sinh(t) = (e^t - e^{-t})/2$ and that $\cosh^2(t) - \sinh^2(t) = 1$.

- (4 points) Give the curvature formula $\kappa(t)$ in general and compute it.
- (4 points) Give the torsion formula $\tau(t)$ in general and compute it.
- (4 points) Give the arc length formula in general and compute it.
- (2 points) Define "simple curve". Is this curve simple?
- (2 points) What does "regular curve" mean? Is this curve regular?
- (2 points) What does "closed curve" mean? Is this curve closed?
- (2 points) Define what "arc length parametrization" means. Could this curve be parametrized by arc length? Yes, or no?

Problem B (20 points): We now look at the planar curve

$$r(t) = \begin{bmatrix} \cos(t) \\ \sin(2t) \end{bmatrix}$$

with $t \in [0, 2\pi)$. It looks like a figure 8.

- a) (4 points) What is its signed curvature $\kappa(t)$?
- b) (4 points) What is its rotation index ρ ?
- c) (4 points) Would knowing $\kappa(t)$ determine the shape of the curve? Why?
- d) (2 points) Is the curve simple? Is the curve closed?
- e) (2 points) State the Hopf Umlaufsatz. Does it apply for this curve?
- f) (2 points) State the 4 vertex theorem. Does it apply for this curve?
- g) (2 points) Define what a vertex is. Find a vertex for the curve.

Problem C (20 points): Define the surface with parametrization

$$r(u, v) = \begin{bmatrix} u \\ v \\ f(u, v) \end{bmatrix} = \begin{bmatrix} u \\ v \\ u^2 - 2u + v^2 \end{bmatrix}.$$

- a) (4 points) Write down the Jacobian matrix dr .
- b) (4 points) Write down the first fundamental form I .
- c) (4 points) Write down the second fundamental form II .
- d) (4 points) Find the surface area for $R = \{u^2 + v^2 \leq 4\}$.
- e) (4 points) Verify that the curvature at $(1, 0)$ agrees with the discriminant $D = f_{uu}f_{vv} - f_{uv}^2$ of the critical point $(u, v) = (1, 0)$.