DIFFERENTIAL GEOMETRY

MATH 136

Lecture 15: Theorema Egregium

14.1. In 1827, Karl Friedrich Gauss proved the **"theorema egregium"**. Is curvature determined by distance measurements within the geometry alone, without reference to the ambient space $\mathbb{R}^3$ in which M is embedded? The answer is yes:

Theorem 1 (Theorema Egegium)**.** *Gaussian curvature K is determined by I.*

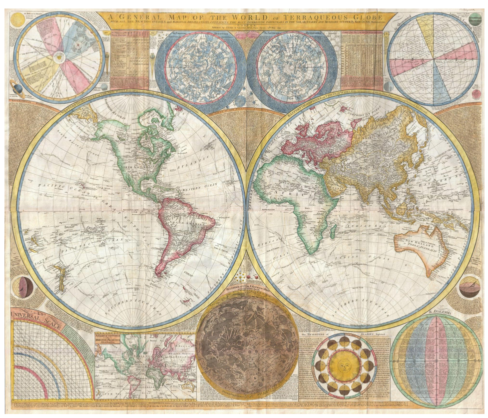

FIGURE 1. 1794 Map by mathematician Sam Dunn, when Gauss was 17.

14.2. This implies that if two spaces have different curvature, they can not be isometric. That curvature in a surface M can be expressed using the first fundamental form I is not surprising, given that the first fundamental form is used for all distance measurements in M using geodesics curves like light beams. But curvature has been defined as $\det(A)$, using the **shape operator** $A = I^{-1}II$. It invoked both the first and second fundamental form II that made use of the normal vector n in $\mathbb{R}^3$.

14.3. Last time, we saw that we can assign an **orthonormal frame field** $\{z, w\}$ on M which only depends on the first fundamental form. This produces an orthonormal frame $\{z, w, n\}$ at every point $p \in M \subset \mathbb{R}^3$. This "frame field" on M is similar to the Frenet frame field $\{T, N, B\} = \{e_1, e_2, e_3\}$ on a curve, where the Frenet equations told how the frame field moves with time. We are interested in how the field changes when we change u and v. The mathematics is very similar to what we did for curves just that we have now two variables u, v rather than only one variable t. We will need the following formulas to relate the curl of $X = [P, Q] = [z \cdot w_u, z \cdot w_v]$ with curvature.

Lemma 1 (Moving frame lemma).

$$\begin{array}{rcl} z_u &=& (z_u \cdot w)w + (z_u \cdot n)n \\ z_v &=& (z_v \cdot w)w + (z_v \cdot n)n \\ w_u &=& (w_u \cdot z)z + (w_u \cdot n)n \\ w_v &=& (w_v \cdot z)z + (w_v \cdot n)n \end{array}$$

Proof. We expand each of the vectors z_u, z_v, w_u, w_v in the $\{z, w, n\}$ basis:

$$\begin{array}{rcl} z_u &=& (z_u \cdot z)z + (z_u \cdot w)w + (z_u \cdot n)n \\ z_v &=& (z_u \cdot z)z + (z_v \cdot w)w + (z_v \cdot n)n \\ w_u &=& (w_u \cdot z)z + (w_u \cdot w)w + (w_u \cdot n)n \\ w_v &=& (w_v \cdot z)z + (w_v \cdot w)w + (w_v \cdot n)n \end{array}$$

and note that $z \cdot z = 1$ implies $z_u \cdot z = z_v \cdot z = 0$. □

Lemma 2 (X is intrinsic). $X = [P, Q] = [z \cdot w_u, z \cdot w_v]$ *is expressible by* I *alone.*

Proof. **(i)** Let us look at $z \cdot w_u = -(z_u \cdot w)$: We have seen that $z = ar_u, w = br_u + cr_v$, where a, b, c depended only on I. Now $(z_u \cdot w) = (ar_u)_u \cdot w = (a_u r_u + ar_{uu}, br_u + cr_v)$. Multiply out and use $(r_u \cdot r_u) = E$, and $(r_{uu}, r_u) = E_u/2$ and $(r_{uu} \cdot r_v) = F_u - E_v/2$. All these terms involve E, F, G or derivative of those from the first fundamental form I.
(ii) Now do the computation for the second coordinate $z \cdot w_v$. Follow the same steps. You do that in the homework. □

Lemma 3 (Curvature is a Curl). $dX = Q_u - P_v = (z \cdot w_v)_u - (z \cdot w_u)_v = K\sqrt{\det(g)}$.

Proof. The wall is climbed in three pitches:
(Pitch i) $\boxed{(z \cdot w_v)_u - (z \cdot w_u)_v = z_u \cdot w_v - z_v \cdot w_u.}$
Proof. Use the product rule and Clairaut's result $z_{uv} = z_{vu}$.
(Pitch ii) $\boxed{z_u \cdot w_v - z_v \cdot w_u = (n_u \times n_v) \cdot n}$
Proof: Use the **moving frame lemma**, and an identity from the Frenet lecture, as well as $(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$ to get

$$\begin{array}{rcl} z_u \cdot w_v - z_v \cdot w_u &=& (n_u \cdot z)(n \cdot w_v) - (w_u \cdot n)(z_v \cdot n) \\ &=& (n \cdot z_u)(n_v \cdot w) - (w \cdot n_u)(z \cdot n_v) \\ &=& (n_u \times n_v, z \times w) = (n_u \times n_v, n) \; . \end{array}$$

(Pitch iii) $\boxed{(n_u \times n_v) \cdot n = K\sqrt{\det(I)}}$
Proof. The left hand side is (remember $drA = -dn$ defined the shape operator A), $((A_{11}r_u + A_{21}r_v) \times (A_{12}r_u + A_{22}r_v)) \cdot n = \det(A)|r_u \times r_v| = \det(A)\sqrt{\det(I)} = K\sqrt{\det(I)}$. □

14.4. The "Theorema Egregium" is proven: the 1-form X is intrinsic. So, the curl dX and also $K(x)$ are intrinsic. One can "shoot down" the Theorema Egregium also by expressing K in terms of the intrinsic Γ_{ijk}. The computations here will help however in the proof of Gauss-Bonnet.

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