PROBABILITY THEORY

MATH 154

Unit 17: Recurrence

17.1. Fix a probability space $(\Omega, \mathcal{A}, \mathrm{P})$. A map $T : \Omega \to \Omega$ is **measurable** if $T^{-1}(A) \in \mathcal{A}$ for every $A \in \mathcal{A}$. It is **measure preserving** if $\mathrm{P}[T^{-1}(A)] = \mathrm{P}[A]$, $\forall A \in \mathcal{A}$. If T is invertible and T^{-1} is measure preserving too, T is an **automorphism** of $(\Omega, \mathcal{A}, \mathrm{P})$. Automorphisms form a group. For example, on a finite probability space with $\mathrm{P}[A] = |A|/|\Omega|$, the automorphisms are permutations and the ergodic ones are cyclic.

17.2. Probability spaces with measure preserving maps as **morphisms** form a nice **category**. Poincaré proved in 1890:

Theorem 1 (Poincaré recurrence). *Given an automorphism T and $A \in \mathcal{A}$ with $\mathrm{P}[A] > 0$, there exists n such that $\mathrm{P}[T^n(A) \cap A] > 0$.*

Proof. If not, then $A_n = T^n(A)$ is a distinct set of events. Let $n > 1/\mathrm{P}[A]$ be an integer. Use finite additivity to see $1 = \mathrm{P}[\Omega] \geq \mathrm{P}[\bigcup_{k=1}^n A_k] = \sum_{k=1}^n \mathrm{P}[A_k] = n\mathrm{P}[A] > 1$ which is a contradiction. □

17.3. Recall that T is called **ergodic** if every fixed point $T(A) = A$ has probability 0 or 1. In short, this means $\mathrm{P}[A + T(A)] = 0 \Rightarrow \mathrm{P}[A]^2 = \mathrm{P}[A]$. If we take a single random variable X, it defines a sequence $X_n(\omega) = X(T^n(\omega))$ of random variables, where $T^n(\omega) = T(T(\dots T(\omega)))$ is the n'th iterate. If T is ergodic and $X = 1_A$ then $\mathrm{E}[S_n]/n \to \mathrm{E}[X] = \mathrm{P}[A]$. The ergodic theorem tells us that the frequency of the number of times that we hit A is the same than the probability of A. The catch phrase is that **"space average agrees with time average"**.

17.4. Let $\Omega = \{|z| = 1\} \subset \mathbb{C}$ be the unit circle in the complex plane equipped with the probability measure $\mathrm{P}[\mathrm{Arg}(z) \in [a, b]] = (b - a)/(2\pi)$ for $0 < a < b < 2\pi$ and the Borel σ-algebra $\mathcal{A}$. If $w = e^{2\pi i\alpha}$ is a complex number of length 1, then the rotation $T(z) = wz$ defines a measure preserving transformation on $(\Omega, \mathcal{A}, \mathrm{P})$. It is invertible with inverse $T^{-1}(z) = z/w$. This system is called the **Kronecker system**. It can be written additively as $\theta \to \theta + \alpha \bmod 2\pi$.

Theorem 2. *If α is irrational, then the Kronecker system is ergodic.*

Proof. With $z = e^{2\pi ix}$, one can write a random variable $X \in \mathcal{L}^2$ on Ω as a Fourier series $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ with $a_n = \mathrm{E}[z^n X]$. We can write $f = f_0 + f_+ + f_-$, where $f_+ = \sum_{n=1}^{\infty} a_n z^n$ is analytic in $|z| < 1$ and $f_- = \sum_{n=1}^{\infty} a_n z^{-n}$ is analytic in $|z| > 1$ and f_0 is constant. By doing the same decomposition for $f(T(z)) = \sum_{n=-\infty}^{\infty} a_n w^n z^n$, we see that $f_+ = \sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} a_n w^n z^n$. But these are the Taylor expansions of

$f_+ = f_+(T)$ and so $a_n = a_n w^n$. Because $w^n \neq 1$ for irrational α, we deduce that $a_n = 0$ for $n \geq 1$. Similarly, one derives $a_n = 0$ for $n \leq -1$. Therefore $f(z) = a_0$, meaning f is constant. $\square$

17.5. It follows that for every number $x \in [0,1]$ and every irrational α and every $\epsilon > 0$, there exists n such that $|n\alpha - x| < \epsilon$.

17.6. The transformation $T(z) = z^2$ on the same probability space as in the previous example is also measure preserving. Note that $\mathrm{P}[T(A)] = 2\mathrm{P}[A]$ but $\mathrm{P}[T^{-1}(A)] = \mathrm{P}[A]$ for all $A \in \mathcal{A}$. The map is measure preserving, but it is **not invertible**.

Theorem 3. *The squaring transformation $T(z) = z^2$ on the circle is ergodic.*

Proof. A Fourier argument shows it again: T preserves again the decomposition of f into three analytic functions $f = f_- + f_0 + f_+$ so that $f(T(z)) = \sum_{n=-\infty}^{\infty} a_n z^{2n} = \sum_{n=-\infty}^{\infty} a_n z^n$ implies $\sum_{n=1}^{\infty} a_n z^{2n} = \sum_{n=1}^{\infty} a_n z^n$. Comparing Taylor coefficients of this identity for analytic functions shows $a_n = 0$ for odd n because the left hand side has zero Taylor coefficients for odd powers of z. But because for even $n = 2^l k$ with odd k, we have $a_n = a_{2^l k} = a_{2^{l-1}k} = \cdots = a_k = 0$, all coefficients $a_k = 0$ for $k \geq 1$. Similarly, one sees $a_k = 0$ for $k \leq -1$. $\square$

17.7. The single angle random variable $X(x) = \arg(x)$ on Ω, produces a sequence of random variables $X_n(x) = X(T^n(x))$. The squaring system is conjugated to the shift $S(x)_n = x_{n+1}$ on the product probability space $(\{0,1\}^{\mathbb{N}}, \mathcal{B}, \mathrm{P})$. The conjugating map is $\phi(x) = e^{2\pi x_i/2^i} \in \Omega$. We have $\phi(S(x)) = T(\phi(x))$.

17.8. If A is an event with $\mathrm{P}[A] > 0$ and T is a measure preserving automorphism, we can define a new transformation $T_A(x) = T^{n_A(x)}(x)$, where $n_A(x)$ is the **return time**, the smallest $n > 0$ with $T^n(x) \in A$. By Poincaré recurrence, the random variable $n_A(x)$ is finite for almost all $x \in A$. We can look at the **conditional probability space** $(A, \mathcal{A} \cap A, \mathrm{P}/\mathrm{P}[A])$ and the **induced dynamical system** T_A.

Theorem 4. *T_A is an automorphism of $(A, \mathcal{A} \cap A, \mathrm{P}/\mathrm{P}[A])$. It is ergodic if T is ergodic.*

Proof. (i) T_A is measure preserving. Decompose $A = \bigcup_{k=1}^{\infty} A_k$ with $A_k = \{n_A = k\}$. Now $T_A(x) = T^k(x)$ for $x \in A_k$. Given $B \in \mathcal{A} \cap A$ define $B_k = B \cap A_k$ so that $B = \bigcup_k B_k$. $\mathrm{P}[T_A^{-1}(B)] = \mathrm{P}[T_A^{-1}(\bigcup_k B_k)] = \mathrm{P}[\bigcup_k T_A^{-1}(B_k)] = \mathrm{P}[\bigcup_k T^{-k} B_k] = \sum_k \mathrm{P}[T^{-k}(B_k)] = \sum_k \mathrm{P}[B_k] = \mathrm{P}[\bigcup_k B_k] = \mathrm{P}[B]$. (ii) If T is ergodic then T_A is ergodic. Proof: use contradiction. If $T_A(B) = B$ has $\mathrm{P}[B] < \mathrm{P}[A]$ then $C = \bigcup_k T^{-k}(B)$ is T invariant with $\mathrm{P}[C] < 1$ and T is not ergodic. $\square$

17.9. Note that it is possible that T_A is ergodic but T is not ergodic. Kakutani noticed that if $\bigcup_k T^k A = \Omega$, then T_A is ergodic if and only if T is ergodic.

17.10. If T is a measure preserving transformation on a probability space, we can look at the longer incidences $A \cap T^{-n}(A) \cap \cdots \cap T^{-kn}(A)$. Fürstenberg showed in 1977:

Theorem 5 (Multiple recurrence theorem)**.** *For any $A \in \mathcal{A}$ with $\mathrm{P}[A] > 0$, there exists n such that $\mathrm{P}[A \cap T^{-n}(A) \cap \cdots \cap T^{-kn}(A) \cap A] > 0$.*

It implies the **van der Waerden theorem** telling that any r coloring of the integers contains a color which contains arbitrary large arithmetic progressions.