PROBABILITY THEORY

MATH 154

Unit 10: Problem Seminar

10.1. Finite probability taps into **combinatorics**. Here are some examples:

$n!$	There are $5! = 120$ possible ways to redistribute 5 coats to 5 people.
$\frac{n!}{n_1!\cdots n_k!}$	With $\{A, A, B, B, B, B, E, U, U, U\}$, form $10!/(2!4!1!3!)$ ten letter words.
$\frac{n!}{(n-k)!}$	10 people can sit $10!/6! = 10 * 9 * 8 * 7$ possible ways on 4 chairs.
n^k	There are 6^{10} possible ways to throw 10 dices.
$\frac{n!}{(n-k)!k!}$	There are $52!/(5!47!)$ hands of 5 cards a deck of 52.

Problem 1: a) You pick 7 cards at random from a deck of 81 cards of the game "set". What is the probability that all of them are red (27 are red)?
b) You throw a dice 7 times. What is the probability that all 7 numbers are even?
c) Your gym lock consist of 3 different numbers from 1 to 40. Having forgotten the number, you try a random combinations. What is the probability to open it?
d) How many bijective functions $X \to X$ are there on $X = \{1, 2, 3, 4, 5, 6, 7\}$?
e) How many functions $X \to Y$ are there from $X = \{1, 2, 3, 4, 5, 6, 7\}$ to $\{1, 2\}$?

Solution:
a) The probability space has $B(81, 7) = 81!/(7!74!)$ elements. The set of all 7 card hands which are red has $B(27, 7) = 27!/(7!20!)$ elements. The answer therefore is $B(27, 7)/B(81, 7)$.
b) The probability space has 6^7 elements. The event that all dice show even numbers has 3^7 elements. The answer is $3^7/6^7$.
c) There are $40*39*37$ possible cases. The probability to hit the right one is $1/(40*39*37)$.
d) $7! = 5040$.
e) There are 2^7 possibilities to assign either 1 or 2 to a number.

10.2. The **laboratory** Ω is a set of experiments. The **σ-algebra** $\mathcal{A}$ consists of events. A σ-algebra is a Boolean algebra which allows to perform **countably** many operations.

$A \cdot B = A \cap B = \{\omega \in \Omega \mid \omega \in A \text{ and } \omega \in B\}$	"Both events A and B happen"
$A \cup B = \{\omega \in \Omega \mid \omega \in A \text{ or } \omega \in B\}$	"Either A or B happens"
$A + B = A\Delta B = \{\omega \in \Omega \mid \omega \in A \text{ xor } \omega \in B\}$	"One of the events A or B happens"
$A \setminus B = \{\omega \in \Omega \mid \omega \in A \text{ but not } \omega \in B\}$	"A but not B happens"
$A^c = \{\omega \in \Omega \mid \omega \notin A\}$	"A does not happen"
$\bigcap_n A_n = \{\omega \in \Omega \mid \omega \in A_n, \text{ for all } n\}$	"All events A_n happen"
$\bigcup_n A_n = \{\omega \in \Omega \mid \omega \in A_n, \text{ for at least one } n\}$	"At least one event A_n happens"

Problem 2: We use the notation $A \cdot B = A \cap B$ and $A + B = A\Delta B$ and $1 = \Omega$ and $0 = \emptyset$ in the Boolean algebra $\mathcal{P} = 2^\Omega$ of all subsets of Ω.
a) Draw the Venn diagram picture proving that $A(B - C) = AB - AC$.
b) Simplify $(5A + 2)(3A^2 + A - 1)$.
c) Write $A^n = A \cdot A \cdot A \cdots A$ for the n'th power. Simplify $(A - 1)(A + A^2 + A^3)$.
d) Why is $(1 + A)^3 = 1 + A$?
e) Show that $A \cup B = A + B + AB$.

Solution:
a) Note that $B - C$ is $B + C$ the symmetric difference because $2C = C + C = 0$. The picture is therefore the same Venn diagram what we have in the notes.
b) Use $A^2 = A$ and $2A = 0$ and $2 = 1+1 = 0$. So $(5A+2)(3A^2+A-1) = A(A+A-1) = A$.
c) From $A^2 = A^3 = A$ we see that the second factor is A. The first factor is A^c. The result is 0. An other possibility is to simplify first $A^4 - A = A - A = 0$.
d) $1 + A = A^c$ and $(A^c)^3 = A^c$.

10.3. If B has positive probability, then $\mathrm{P}[A|B] = \mathrm{P}[A \cap B]/\mathrm{P}[B]$ is called the **conditional probability** of A under the condition that event B takes place.

Problem 3: a) If the probability that a student is sick at a given day is 1 percent and the probability that a student has an exam at a given day is 5 percent. Suppose that 6 percent of the students with exams are ill. What is the probability that an ill student has an exam on a given day?
b) Suppose that A, B are subsets of a sample space with a probability function P. We know that $\mathrm{P}[A] = 4/5$ and $\mathrm{P}[B] = 3/5$. Explain why $\mathrm{P}[B|A]$ is at least $1/2$.

Solution:
a) Let A be the event that a student is sick. Let B the event that there is an exam. $\mathrm{P}[A] = 1/100$ and $\mathrm{P}[B] = 5/100$ and $\mathrm{P}[A|B] = 6/100$. We know $\mathrm{P}[A \cap B] = (6/100) * (1/100)$. By Bayes theorem $\mathrm{P}[B|A] = (6/5)(1/100)$ which is 1.2 percent.
b) $\mathrm{P}[B|A] = \mathrm{P}[B \cap A]/\mathrm{P}[P[A] \geq (2/5)/(4/5) = 1/2$ because $\mathrm{P}[A \cap B] \geq 2/5$.

10.4. The linear space $\mathcal{L}^2$ has an **inner product** $\mathcal{X} \cdot \mathcal{Y} = \mathrm{E}[XY]$ and so a **length** $|X| = \sqrt{X \cdot X}$. The **standard deviation** of X is the length of **centered random** variable $X - \mathrm{E}[X]$. The **correlation** $-1 \leq \mathrm{Cov}[X,Y]/(\sigma[X]\sigma[Y]) \leq 1$ is $\cos(\alpha)$ and defines an **angle** α between $X - \mathrm{E}[X]$ and $Y - \mathrm{E}[Y]$. If X takes finitely many values (which means $X \in \mathcal{S}$), then $\mathrm{E}[X^n] = \sum_{x \in X(\Omega)} x^n \mathrm{P}[X = x]$. For $X \in \mathcal{L}^n$ with a PDF f, then $\mathrm{E}[X^n] = \int_{\infty}^{\infty} x^n f(x)\, dx$. In the box, c, λ are constants.

$\mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y]$	$\mathrm{E}[\lambda X] = \lambda \mathrm{E}[X]$
$X \leq Y \Rightarrow \mathrm{E}[X] \leq \mathrm{E}[Y]$	$\mathrm{E}[X^2] = 0 \Leftrightarrow X = 0$
$\mathrm{E}[X] = c$ if $X(\omega) = c$	$\mathrm{E}[X - \mathrm{E}[X]] = 0$
$\mathrm{Var}[X] \geq 0$	$\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$
$\mathrm{Var}[\lambda X] = \lambda^2 \mathrm{Var}[X]$	$\mathrm{Cov}[X,X] = \mathrm{Var}[X]$

Problem 4: Let $([-\pi,\pi], \mathcal{B}, \mathrm{P})$ be the Lebesgue probability space, where $\mathrm{P}[[a,b)] = (b-a)/(2\pi)$ on the π system of all half open intervals on $\Omega = [-\pi,\pi]$.
a) Which theorem assures that the measure P exists?
b) Let $X(x) = \sin(3x), Y(x) = \cos(3x)$. Compute the $\mathrm{E}[X], \mathrm{E}[Y], \sigma[X], \sigma[Y]$.
c) What is the correlation $\mathrm{Cor}[X,Y] = \mathrm{Cov}[X,Y]/\sigma(X)\sigma(Y)$? Are X, Y independent?

Solution:
a) This is assured by the Carathéodory extension theorem.
b) Both $\mathrm{E}[X] = \mathrm{E}[Y] = 0$ and $\mathrm{E}[X^2] = \mathrm{E}[Y^2] = 1/2$ so that $\sigma[X] = \sigma[Y] = 1/\sqrt{1/2}$.
c) The Covariance is zero as $\int_{-\pi}^{\pi} \sin(3x)\cos(3x)\, dx = 0$. The two random variables are uncorrelated.
The random variables are dependent however, as $X^2 + Y^2 = 1$.

10.5. Assume X has a probability density $F' = f$ then $\mathrm{E}[X^n] = \int x^n f(x)\, dx$ and $\mathrm{E}[e^{tX}] = \int e^{tx} f(x)\, dx$, $\mathrm{E}[e^{itX}] = \int e^{itx} f(x)\, dx$. Now form $\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$ etc.

Problem 5: The PDF $f(x) = \frac{2}{\pi\sqrt{1-x^2}}$ is supported on $[-1,1]$.
a) Compute the cumulative distribution function $F(x) = \int_{-\infty}^{x} f(t)\, dt = \int_{-1}^{x} f(t)\, dt$.
b) Write down the integral for the moment generating function $M_X(t)$.
c) Express the variance in terms of $M_X'(0)$ and $M_X''(0)$.
d) Relate $M_X(t)$ and $M_Y(t)$ and $M_{X+Y}(t)$ for independent random variables! State the same law for the characteristic function $\phi_X(t), \phi_Y(t)$.

Solution:
a) $\frac{2}{\pi}\int_{-1}^{x} \frac{1}{\sqrt{1-t^2}}\, dt = 1 + \frac{2}{\pi}\arcsin(x)$.
b) $\frac{2}{\pi}\int_{-1}^{1} e^{tx} \frac{1}{\sqrt{1-t^2}}\, dt$ and $\frac{2}{\pi}\int_{-1}^{1} e^{itx} \frac{1}{\sqrt{1-t^2}}\, dt$.
c) $\mathrm{E}[X^2] - E[X]^2 = \frac{d^2}{dt^2} M_X(t)|_{t=0} - (\frac{d}{dt} M_X(t)|_{t=0})^2$.
d) $M_X(t)M_Y(t) = M_{X+Y}(t)$ and $\phi_X(t)\phi_Y(t) = \phi_{X+Y}(t)$.

10.6. Assume X is a random variable taking a finite or countable number of values $P[X = x_k] = p_k$. Then $\mathrm{E}[X^n] = \sum_k x_k^n p_k$, $\mathrm{E}[e^{tX}] = \sum_k e^{tx_k} p_k$ and $\mathrm{E}[e^{itX}] = \sum_k e^{itx_k} p_k$.

Problem 6: Assume X is a random variable that takes the value 3 with probability 1/3 and the value 6 with probability 2/3.
a) Find the expectation $m = \mathrm{E}[X]$ and $n = \mathrm{E}[X^2]$.
b) Are X, X^2 independent? Explain.
c) Write down the characteristic function.

Solution:
a) $m = \mathrm{E}[X] = 3(1/3) + 6(2/3) = 5$ and $n = \mathrm{E}[X^2] = 9(1/3) + 36(2/3) = 27$.
$\mathrm{E}[XX^2] = \mathrm{E}[X^3] = 27/3 + 144 = 153$.
b) No. 153 is not $5 * 27$ so that there is correlation.
c) $e^{it3}/3 + 2e^{it6}/6$.

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