5/6/2011: Final exam

Your Name:

- Please enter your name in the above box. You have to do this exam on your own.
- You can write the solutions on your own paper. Make sure you write your name on each page and staple it together at the end.
- Please write neatly Answers which are illegible for the grader can not be given credit.
- Unless stated, give computations, details reasons. The suggested exam time is Friday, the 6th of May.
- No calculators, computers, or other electronic aids are allowed. There is no time limit but 3 hours should suffice. Bring your exam to my office in 434 on Monday May 9th, 2011 until noon 12. I need to get the paper in Person this can be done on Friday at 4 or Monday between 10 and 12 AM.
- The exam is open book in the sense that you can use any of the distributed class lecture notes and any of your handwritten notes but no other books or text resources.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points)

1) T F A random variable on the probability space $\Omega = \{1, 2, \ldots, 10\}$ satisfying $P[X = k] = 10!/(k!(10-k)!2^{10})$ has the expectation $E[X] = 10/2$.

Solution:
This is the definition.

2) T F There is a system of linear equations $A\vec{x} = \vec{b}$ which has exactly 2 solutions.

Solution:
Consistent means that there is one solution. In general, there is either no, 1 or infinitely many solutions.

3) T F If the sum of all geometric multiplicities of a 3×3 matrix A is equal to 3, then A is diagonalizable.

Solution:
We have then an eigenbasis.

4) T F If a system of linear equations $Ax = b$ has two different solutions x, then the nullety of A is positive.

Solution:
There is a kernel then given by the difference of the two solutions.

5) T F If a matrix A is invertible then its row reduced echelon form is also invertible.

Solution:
The kernel does not change.

6) T F All symmetric matrices are invertible.

Solution:
Take the zero matrix.

7) T F All symmetric matrices have simple spectrum in the sense that all of the eigenvalues are different.

Solution:
Take the identity matrix.

8) T F The geometric multiplicity is always larger or equal than the algebraic multiplicity.

Solution:
It is the other way round.

9) T F The Google matrix is a Markov matrix

Solution:
Yes, it is a combination of Markov matrices built in such a way that the sum is Markov too.

10) T F If a 3×3 matrix has positive entries and is symmetric, then all eigenvalues are different.

Solution:
All eigenvalues are real and there is a largest eigenvalue different from others. But the other two can be the same. An example is $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

11) T F If a 2×2 matrix has positive entries, then both eigenvalues are real and different.

Solution:
Yes, now both eigenvalues are real and one is larger than the other.

12) T F If two random variables X, Y are independent then $\sigma[X+Y] = \sigma[X]+\sigma[Y]$.

Solution:
It is the variance which adds up, not the standard deviation.

13) T F An exponentially distributed random variable has positive expectation.

Solution:
By definition.

14) T F The variance of the Cauchy distributed random variable is 1.

Solution:
The variance does not exist.

15) T F If two events A, B satisfy that $\mathrm{P}[A|B] = \mathrm{P}[A]$ then A, B are independent.

Solution:
By definition of conditional expectation

16) T F For the Monty-Hall problem, it does not matter whether you switch or not. The winning chance is the same in both cases.

Solution:
Yes, we covered this in class. You solved the 4 door problem even.

17) T F A random variable on a finite probability space $\{1, 2, 3\}$ can be interpreted as a vector with three components.

Solution:
Yes, this was a main point we wanted to make throughout the course.

18) T F The composition of two reflections in the plane is an orthogonal transformation.

Solution:
Reflections are always orthogonal. The composition of orthogonal transformations is orthogonal

19) T F If $\mathrm{P}[A|B]$ and $\mathrm{P}[A]$ and $\mathrm{P}[B]$ are known, then we can compute $\mathrm{P}[B|A]$.

20) T F If the P-value of an experiment is less than 15 percent, then the null hypothesis is rejected.

21) T F Its not 15 percent.

Problem 2) (10 points)

a) (5 points) Which matrices are diagonalizable? Which ones are Markov matrices?

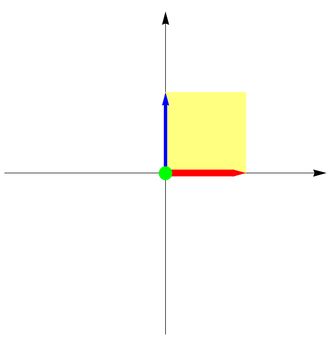
Matrix	diagonalizable	Markov
$\begin{bmatrix} 1/2 & 2/3 \\ 1/2 & 1/3 \end{bmatrix}$		
$\begin{bmatrix} 1/2 & 2/3 \\ 1/2 & 1/2 \end{bmatrix}$		
$\begin{bmatrix} 1/2 & -2/3 \\ 1/2 & -1/3 \end{bmatrix}$		
$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$		
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$		

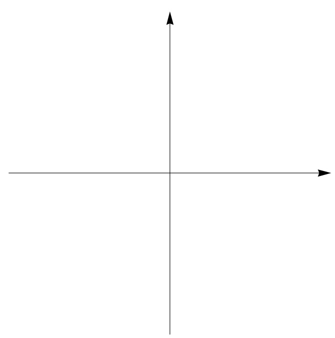
b) (3 points) Match the transformation with the trace and determinant:

trace	det	enter A-F
1	0	
0	-1	
0	1	
1	1/4	
2	1	
-2	1	

label	transformation
A	reflection at the origin
B	rotation by -90 degrees
C	projection onto x-axes
D	dilation by 1/2
E	reflection at a line
F	vertical shear

c) (2 points) Draw the image of the picture to the left under the linear transformation given by the matrix $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$. Make sure to indicate clearly which arrow goes where.





Solution:

Matrix	diagonalizable	Markov
$\begin{bmatrix} 1/2 & 2/3 \\ 1/2 & 1/3 \end{bmatrix}$	x	x
$\begin{bmatrix} 1/2 & 2/3 \\ 1/2 & 1/2 \end{bmatrix}$	x	O
$\begin{bmatrix} 1/2 & -2/3 \\ 1/2 & -1/3 \end{bmatrix}$	x	O
$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$	O	O
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	x	x

b) C,E,B,D,F,A

c) The first basis vector (fat vector) goes to the first column vector $[1,-1]^T$. The second basis vector (thin vector) goes to $[1,2]^T$.

Problem 3) (10 points) Systems of linear equations

Consider the following system of linear equations:

$$\begin{aligned} 3x+y+z &= \lambda x \\ x+3y+z &= \lambda y \\ x+y+3z &= \lambda z \end{aligned}$$

You can interpret the above system as an eigenvalue problem $Ax = \lambda x$ for a 3×3 matrix A. What are the eigenvalues and what are the eigenvectors?

Solution:
The eigenvalues are $5, 2, 2$. The eigenvectors are $[1, 1, 1]^T, [1, -1, 0]^T, [1, 0, -1^T]$. In the original question the λ had been missing and only $[0, 0, 0]^T$ was a soluton.

Problem 4) (10 points) Bayesan statistics

We throw 5 fair dice. Let A be the event that the sum of the first four dice is 5. Let B be the event that the sum of the last two dice is 6.

a) (4 points) Find the probabilities $\mathrm{P}[A], \mathrm{P}[B]$.

b) (3 points) Find the conditional probability $\mathrm{P}[B|A]$.

c) (3 points) What is the conditional probability $\mathrm{P}[A|B]$?

Solution:
We use the Bayes formula $\mathrm{P}[A|B] = \frac{\mathrm{P}[B|A] \cdot P[A]}{\mathrm{P}[B]}$. and know $\mathrm{P}[A] = 4/6^4$ and $\mathrm{P}[B] = 5/6^2$ and $\mathrm{P}[B|A] = 4/24 = 1/6$ and $\mathrm{P}[A|B] = \mathrm{P}[B|A]\mathrm{P}[A]/\mathrm{P}[B] = (1/6)(4/6^4)/(5/36) = 1/270$.

Problem 5) (10 points) Basis and Image

Define the **solar matrix**

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

a) (4 points) Find a basis for the eigenspace to the eigenvalue 1 that is the kernel of $B = A - I_{10}$.

b) (3 points) Find a basis for the image of B.

c) (3 points) Find the determinant of A.

Solution:
a) $\mathcal{B} = \{\begin{bmatrix} 0,1,-1,0,0,0,0,0,0,0 \end{bmatrix}, \begin{bmatrix} 0,1,0,-1,0,0,0,0,0,0 \end{bmatrix}, \begin{bmatrix} 0,1,0,0,-1,0,0,0,0,0 \end{bmatrix}, [\ 0,1,0,0,0,-$
is a basis for the eigenspace.
b) The first two columns of A are a basis for the image.
c) The diagonal pattern gives 2. Then there are 9 patterns for which 7 diagonal entries are one and a matrix $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ of determinant -1 leading to -9. Together we have $2 - 9 = -7$. It is also possible to solve the problem with Laplace expansion with respect to the last row. We can define solar matrices $L(n)$ for any n and see that $L(n) = -1 + L(n-1)$. Since $L(1) = 2$ we get $L(10) = 2 - 9 = -7$.

Problem 6) (10 points) Inverse and Coordinates

a) (5 points) Verify that if

$$A = \begin{bmatrix} B & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & D \end{bmatrix}$$

where B, C, D are 2×2 matrices, then

$$A^{-1} = \begin{bmatrix} B^{-1} & 0 & 0 \\ 0 & C^{-1} & 0 \\ 0 & 0 & D^{-1} \end{bmatrix}.$$

b) (5 points) Invert the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{bmatrix}.$$

Solution:
a) Multiply

$$A^{-1}A = \begin{bmatrix} B^{-1} & 0 & 0 \\ 0 & C^{-1} & 0 \\ 0 & 0 & D^{-1} \end{bmatrix} \begin{bmatrix} B & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & D \end{bmatrix} = \begin{bmatrix} B^{-1}B & 0 & 0 \\ 0 & C^{-1}C & 0 \\ 0 & 0 & D^{-1}D \end{bmatrix} = \begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_2 \end{bmatrix} .$$

b) We can use part a) to get

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -3 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & -2/2 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3/10 & 1/10 \\ 0 & 0 & 0 & 0 & -1/10 & 3/10 \end{bmatrix} .$$

Problem 7) (10 points) Expectation, Variance, Covariance

We throw 5 fair dice and call X_k the number of eyes of dice k. As usual, the X_k are assumed to be independent. Let $X = X_1 + X_2 + X_3$ denote the sum of the first 3 dice and $Y = X_4 + X_5$ denote the sum of the last two dice. Find the correlation $\text{Cov}[X, Y]$ between the two random variables X, Y.

Solution:
The random variables X, Y are independent so that their correlation is zero. Therefore, $\text{Cov}[X, Y] = 0$. We can also use the linearity of covariance to get

$$\text{Cov}[X_1 + X_2 + X_3, X_4 + X_5] = \text{Cov}[X_1, X_4] + ... + \text{Cov}[X_3, X_5] = 0 .$$

Problem 8) (10 points) Eigenvalues and Eigenvectors

a) (7 points) Find all the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 0 & 4 \\ 4 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 3 \\ 3 & 0 & 0 & 0 & 0 & 4 & 0 \end{bmatrix} .$$

Hint: Write A as $3Q + 4Q^T$ for some orthogonal matrix Q for which you know how to compute the eigenvalues and eigenvectors.

b) (3 points) Find the determinant of A.

Solution:
a)We know the eigenvalues $\lambda_k = e^{2\pi i k/7}$ of Q and so the eigenvalues of A are $3\lambda_k + 4\lambda_k^{-1}$. The eigenvectors of A are the same than the eigenvectors of Q and are $[1, \lambda_k, \lambda_k^2, ..., \lambda_k^6]^T$.
b) The determinant is the product of the eigenvalues which is $\prod_k 3\lambda_k + 4\lambda_k^{-1}$. Alternatively, one can see that there are only 2 patterns both with an even number of upcrossings. The determinant is $3^7 + 4^7$.

Problem 9) (10 points) Determinants

a) (3 points) Find the determinant of the matrix

$$A = \begin{bmatrix} 0 & 6 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 9 \\ 3 & 0 & 0 & 0 \end{bmatrix} .$$

b) (3 points) Find the determinant of the matrix

$$A = \begin{bmatrix} 9 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 9 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 9 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 9 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 9 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 9 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 9 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 9 \end{bmatrix} .$$

c) (4 points) Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ -1 & 0 & 3 & 4 & 5 & 6 & 7 & 8 \\ -1 & -2 & 0 & 4 & 5 & 6 & 7 & 8 \\ -1 & -2 & -3 & 0 & 5 & 6 & 7 & 8 \\ -1 & -2 & -3 & -4 & 0 & 6 & 7 & 8 \\ -1 & -2 & -3 & -4 & -5 & 0 & 7 & 8 \\ -1 & -2 & -3 & -4 & -5 & -6 & 0 & 8 \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & 0 \end{bmatrix} .$$

Please indicate clearly the method you have used and as usual provide details for the computation.

Solution:
a) There is only one patten with 3 up-crossings. The answer is $-3*6*7*9$.
b) $A-8I$ has the eigenvalues $0,0,0,0,0,0,0,9$. The matrix A has the eigenvalues $8,8,8,8,8,8,8,18$. The determinant is the product which is $2\cdot 8^8$.
c) Add the first row to the other rows to get a diagonal matrix which has the determinant $8! = 40'320$.

Problem 10) (10 points) Law of large numbers, Central limit theorem

A dice is not fair and shows

number	probability
1	1/7
2	1/7
3	1/7
4	1/7
5	2/7
6	1/7

a) (4 points) Find the mean m and standard deviation σ of X.

b) (3 points) If X_k shows the number of eyes of the k'th dice, what is $\frac{1}{n}\sum_{k=1}^n X_k$ in the limit $n\to\infty$?

c) (3 points) The random variables

$$\sum_{k=1}^n \frac{(X_k-m)}{\sigma\sqrt{n}}$$

converge. In which sense and to what?

Solution:
a) The mean is $26/7$.
The variance is $136/144$.
b) By the law of large numbers, the limit is $26/7$.
c) By the central limit theorem the limiting distribution is the standard normal distribution.

Problem 11) (10 points) Diagonalizable matrices

Among all 2×2 upper triangular matrices for which the entries are 0 or 1. What is the probability that the matrix is diagonalizable? The probability space is

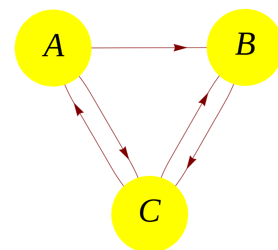
$$\Omega = \{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\}$$

and we assume that we have the uniform distribution on Ω.

Solution:
There are 6 matrices which are diagonalizable. The probability is $6/8 = 3/4$.

Problem 12) (10 points) Markov matrices

You are given a network A,B,C of 3 nodes where all neighboring nodes link to each other but B only links to C and not to A. Find the page rank for each nodes in the case when the damping factor is $d=0.5$.



Solution:
The google matrix is

$$(1/2)\begin{bmatrix} 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1 & 0 \end{bmatrix} + (1/2)\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}/3 = \begin{bmatrix} 2 & 2 & 5 \\ 5 & 2 & 5 \\ 5 & 8 & 2 \end{bmatrix}/12$$

The eigenvector to the eigenvalue 1 normalized so that he largest component is 10 is $[20/3, 50/6, 10]^T$. The page ranks are $6.666, 8.3333, and 10$ or $7, 8, 10$ when rounded. Its no surprise that C has the largest page rank.

Problem 13) (10 points) Important theorems

Complete the placeholders in the following sentences. As usual, we use the notation

$$S_n = X_1 + X_2 + \cdots + X_n$$

and denote by X^* the normalized random variable

$$X^* = (X - \mathrm{E}[X])/\sigma[X] \ .$$

a) (1 point) The **Bayes formula** allows to compute $\mathrm{P}[A|B]$ if $\mathrm{P}[A]$, $\mathrm{P}[B|A]$ and 1 are known.

b) (1 point) The **central limit theorem** assures that S_n^* converges to the 2.

c) (1 point) The **law of large numbers** assures that S_n/n converges to 3.

d) (1 point) **Chebyshev's theorem** assures that $\mathrm{P}[|X - \mathrm{E}[X]| \geq c] \leq$ 4.

e) (1 point) **Perron-Frobenius** assures that a 5 matrix has a unique largest eigenvalue 1.

f) (1 point) The **spectral theorem** tells that a 6 matrix has an orthonormal eigenbasis.

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g) (4 points) These six main results have practical applications. Find a match between applications and results so that each result a) - f) appears exactly once.

Application	enter a)-f)
Estimating P-values of an experiment	
Google page rank	
Monte Carlo integration method	
Counting method in Blackjack	
Change basis to get uncorrelated random variables.	
Estimate the tail distribution $\mathrm{P}[|X| \geq c]$	

Solution:

1	P[B]
2	normal distribution N(0,1)
3	expectation E[X]
4	$Var[X]/c^2$
5	Markov
6	Symmetric

b) b,e,c,a,f,d