Math19b Spring 2011

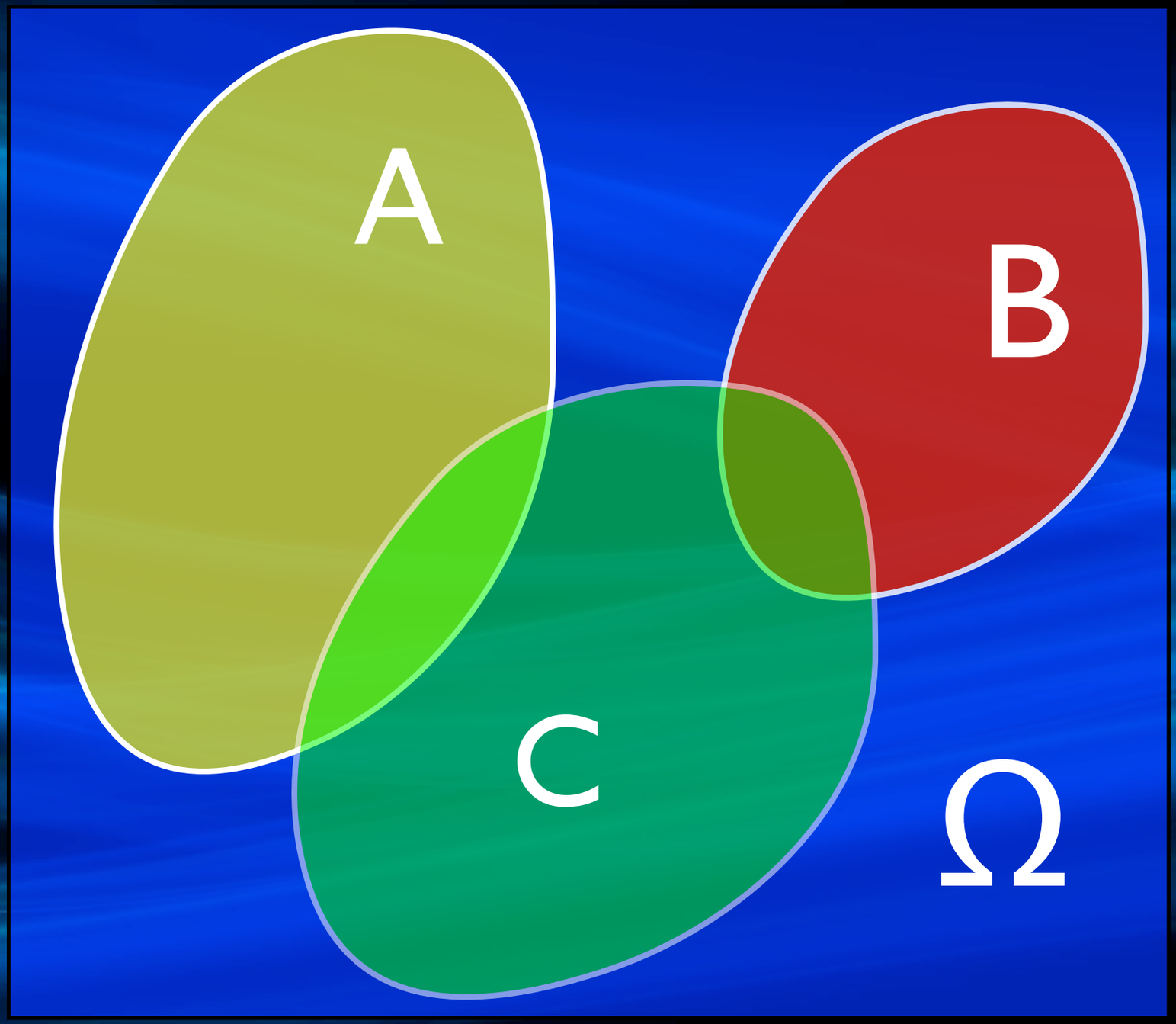
Midterm Review

March 21, 2011

Part I: Probability

- Probability Spaces
- Bayes Formula
- Random variables
- Combinatorics
- Expectation, Variance, Covariance
- Binomial Distribution

Probability Space

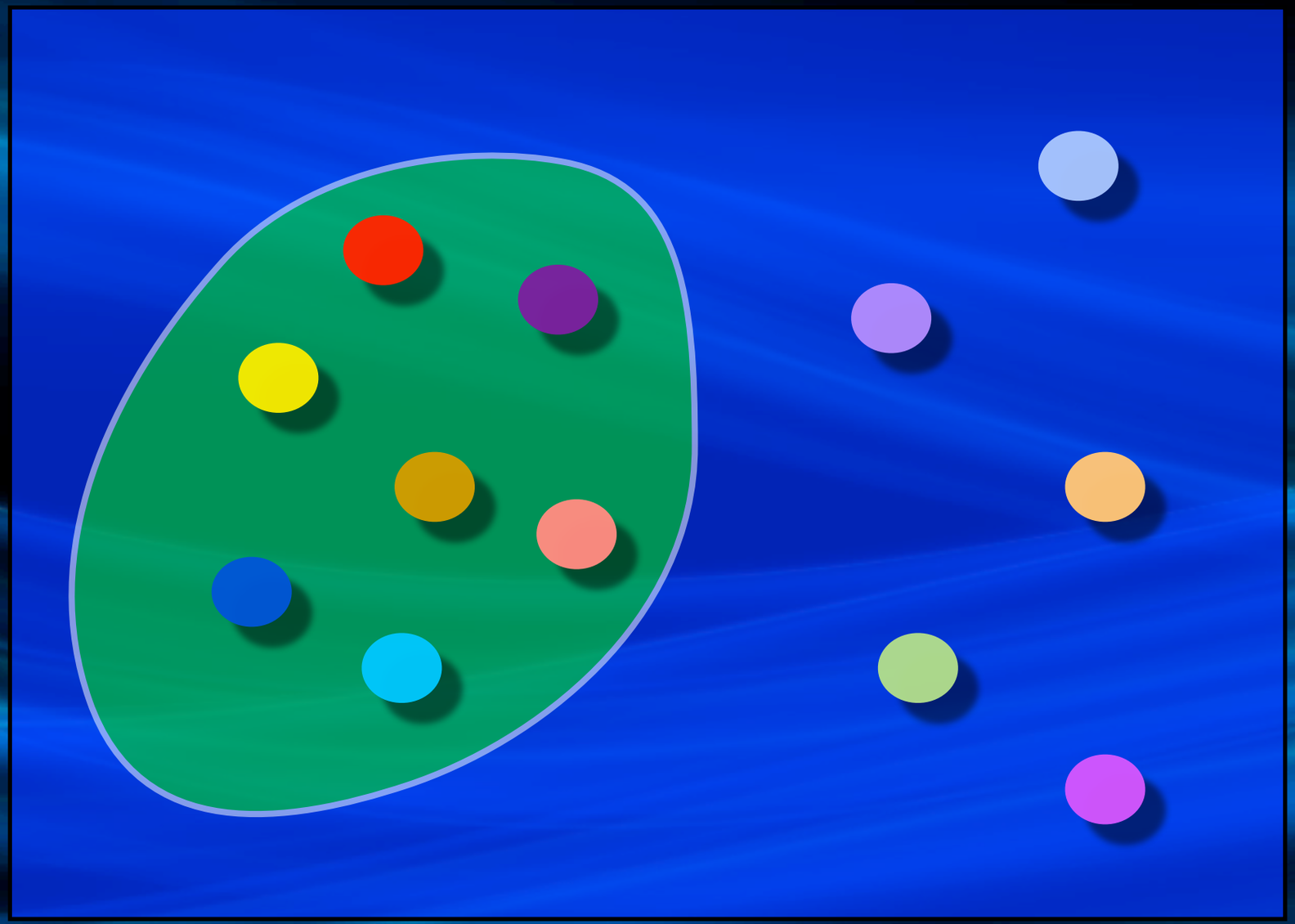


Ω lab

A event

P[A] probability of event A

Finite Probability Space

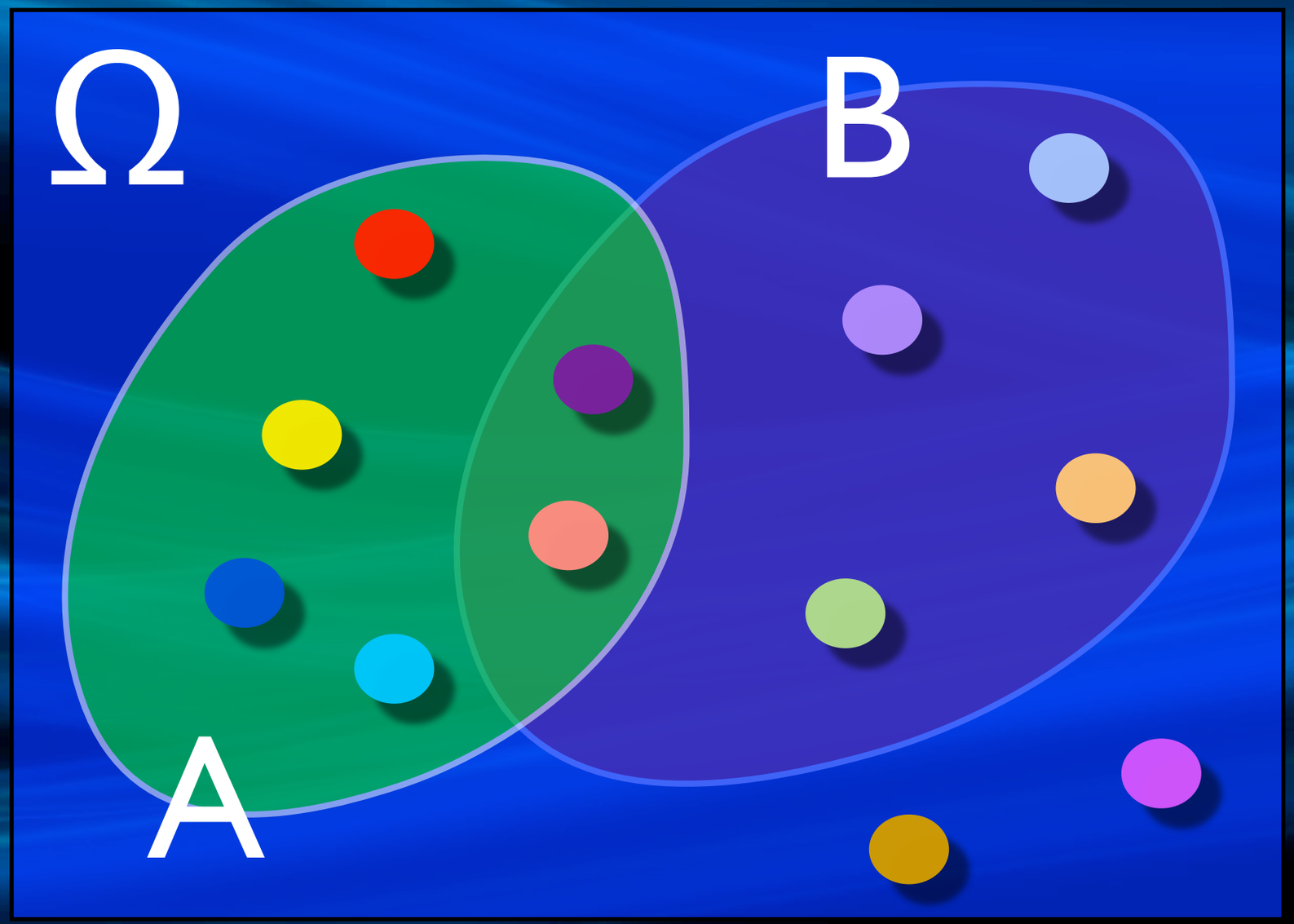


Ω finite

$$P[A] = \frac{|A|}{|\Omega|}$$

$$\frac{\text{good cases}}{\text{all cases}}$$

Conditional Probability



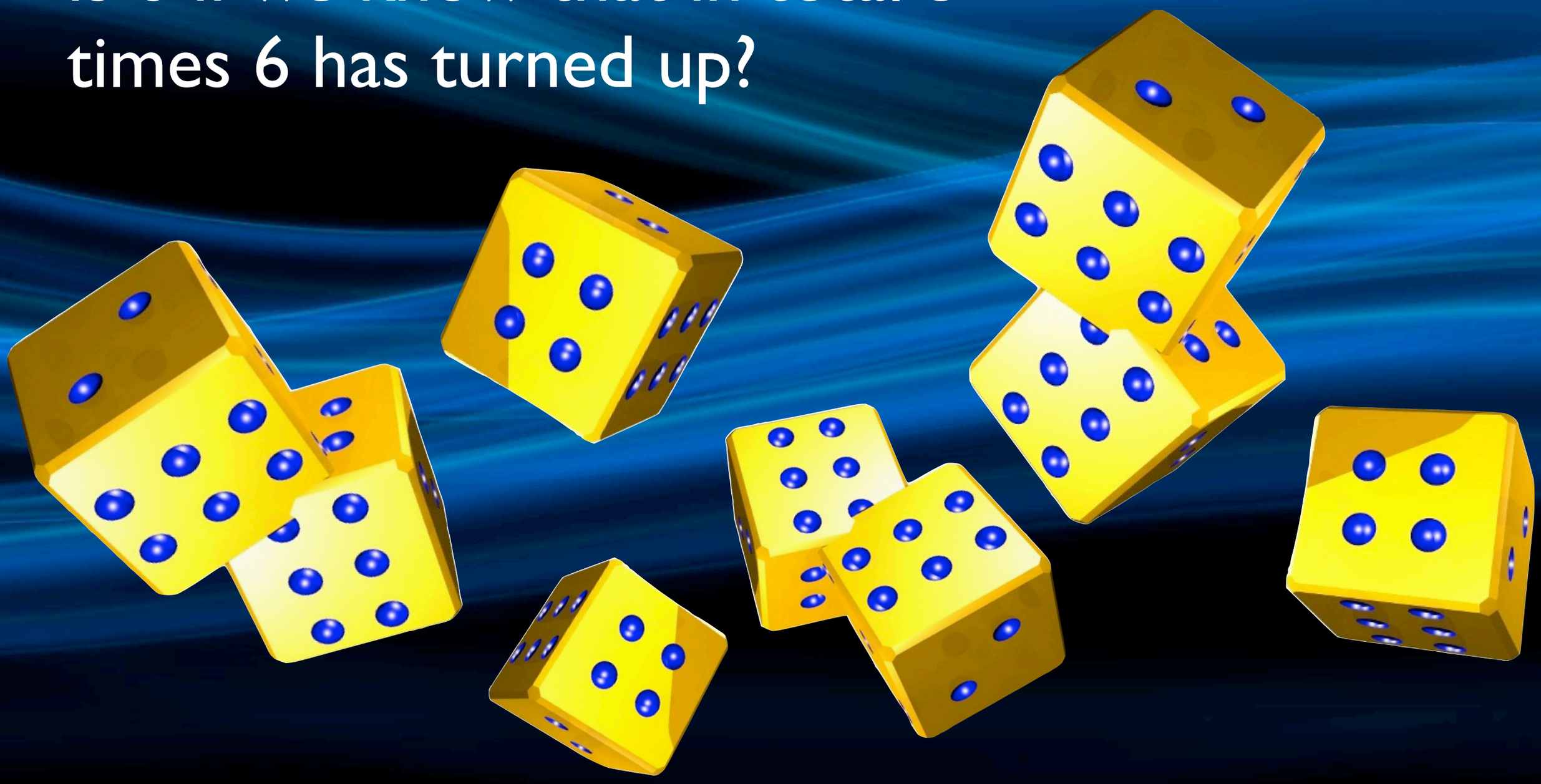
$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$

Bayes Formula

$$P[A|B] = \frac{P[B \mid A] \cdot P[A]}{P[B|A] + P[B|A^c]}$$

Example:

We throw 9 dice. What is the probability that the first dice is 6 if we know that in total 5 times 6 has turned up?



Blackboard problem.

A={first dice =6}
B={5 dice are 6}

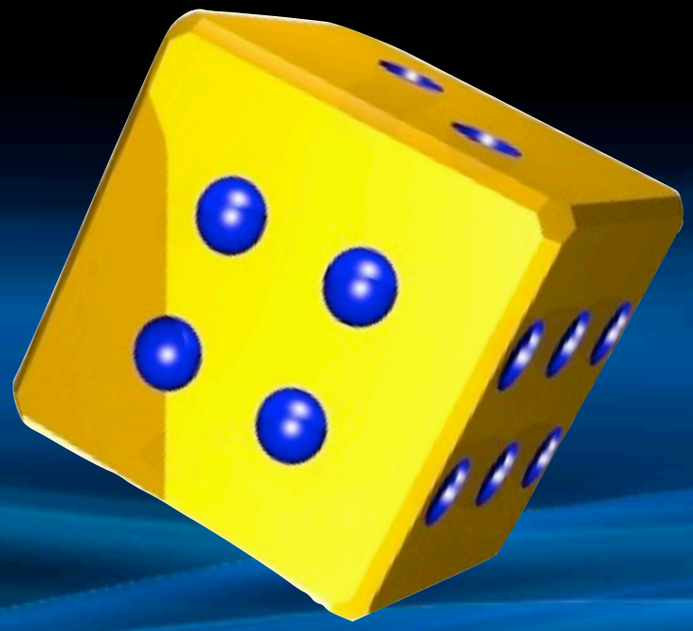
P[A|B] unknown

P[B|A] and P[B|A^c]
are known

P[A] is known

Example (from lecture)

A dice shows k eyes. Throw k coins. All show head. What is the probability that the dice had 6



$$B = \{ \text{all head} \}$$

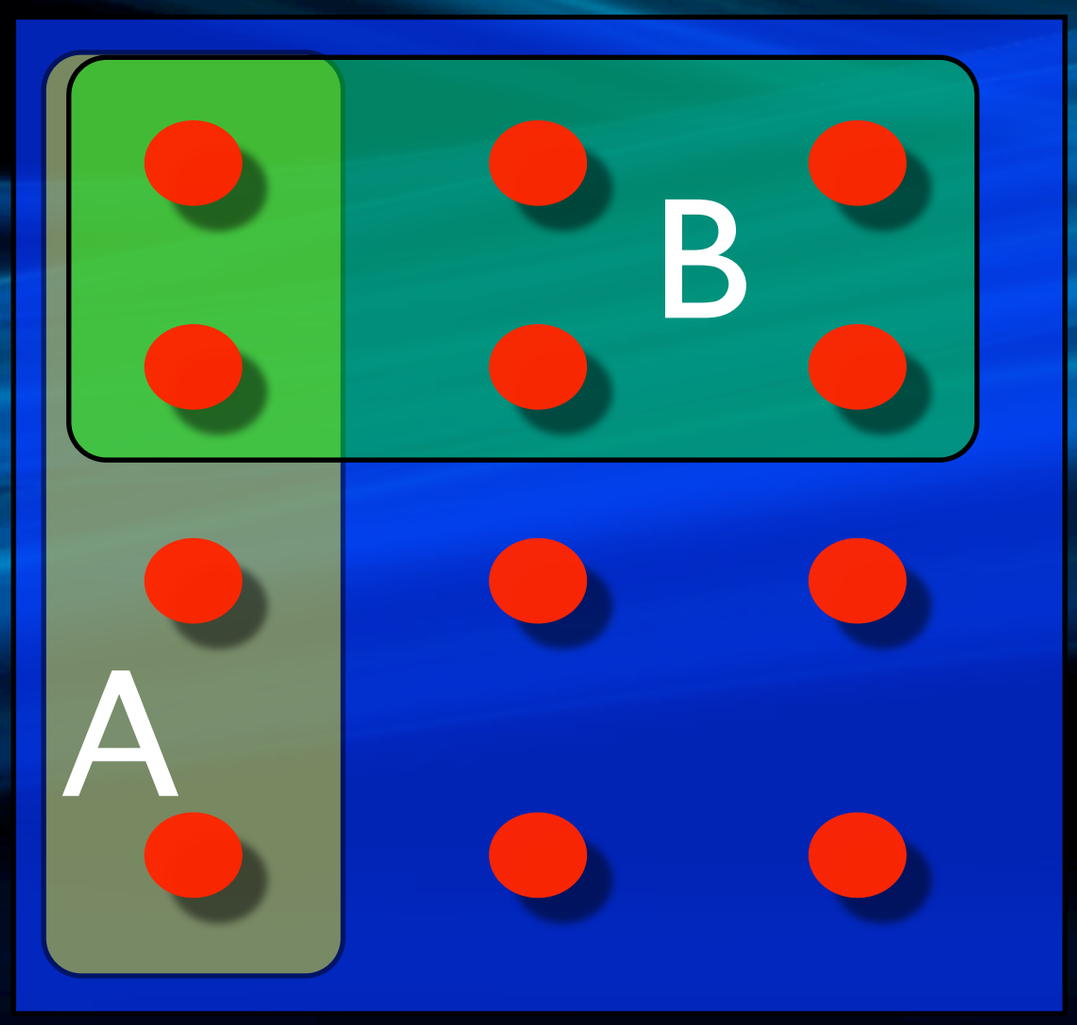
$$A_k = \{ \text{die} = k \}$$

$$P[A_6 | B] = \frac{P[B | A_6]\, P[A_6]}{P[B]}$$

$$P[B] = \sum_k P[B | A_k]\, P[A_k]$$

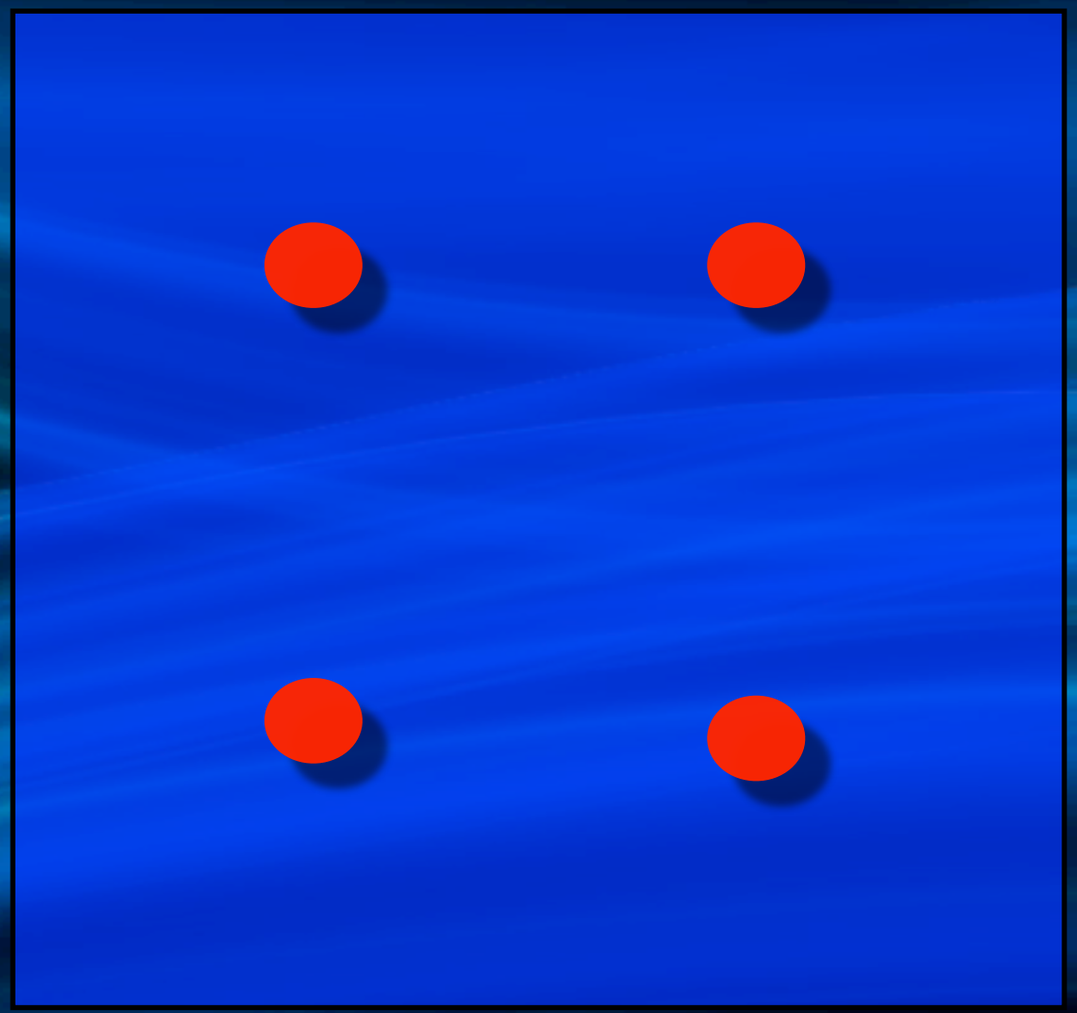
Independent Events

$$P[A \cap B] = P[A]\,P[B]$$



Random Variable

$$X = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 5 \end{bmatrix}$$



$$E[X] = (2+4+1+5)/4 = 3$$

$$Var[X] = E[X^2] - E[X]^2 = (4+16+1+25)/4 - 9$$

Independent vectors
independent random variables

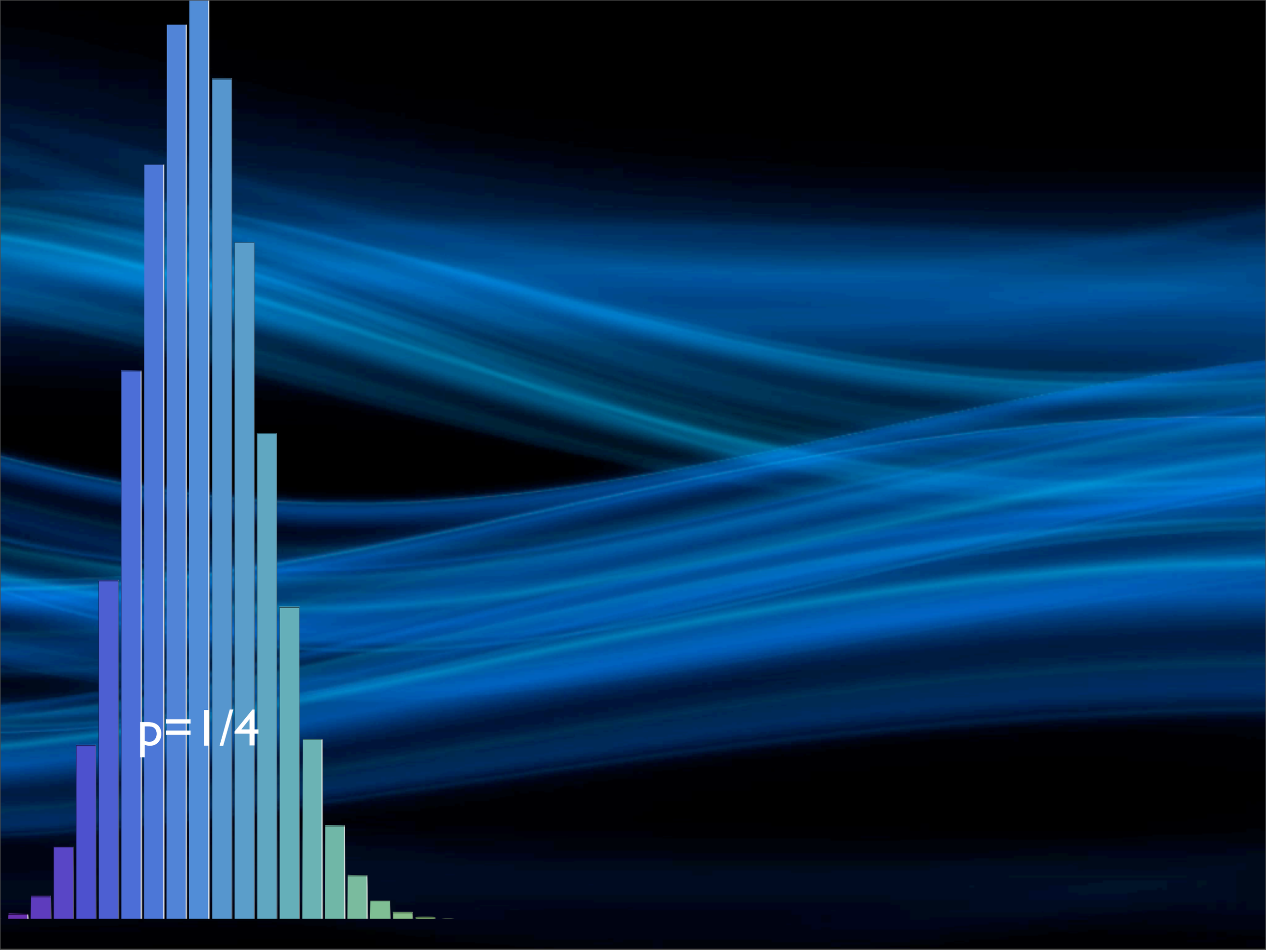
$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad Y = \begin{bmatrix} 2 \\ 1 \\ -1 \\ -2 \end{bmatrix} \quad Z = \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix}$$

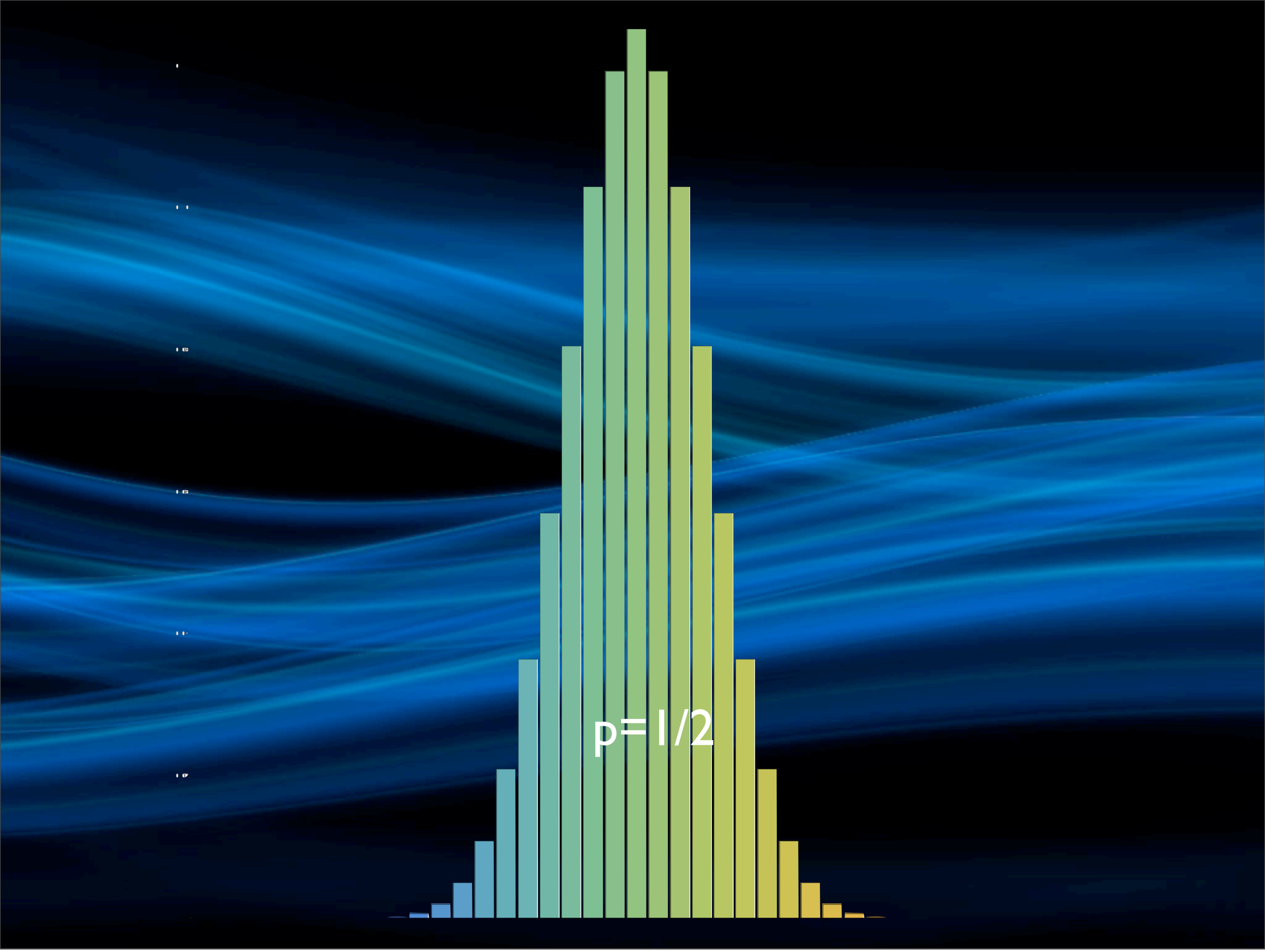
Two of them are independent random variables.
Which ones?

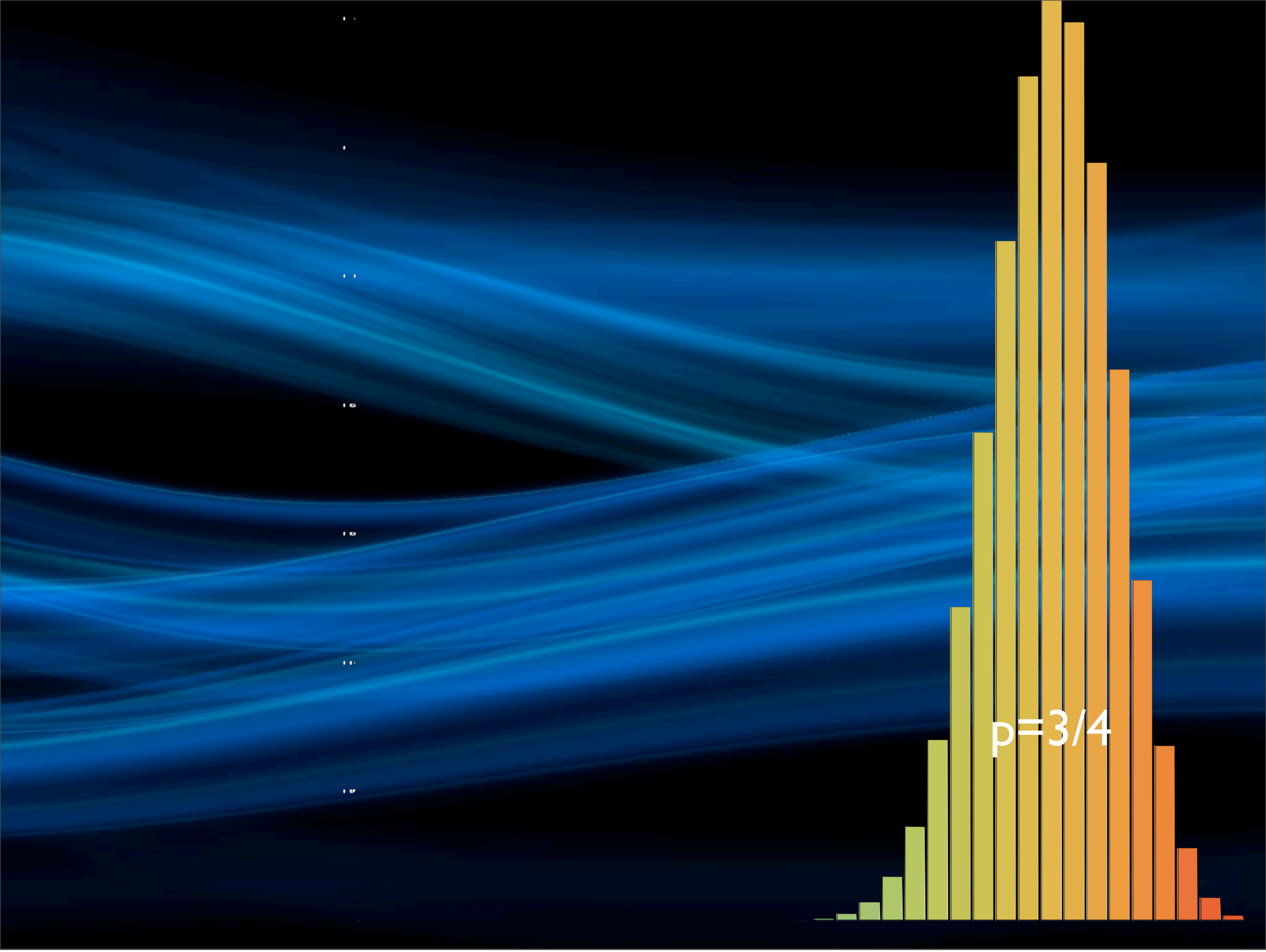
Binomial distribution

$$P[X=k] = \binom{n}{k} p^k (1-p)^{n-k}$$

Probability of k hits


p=1/4


p=1/2


p=3/4

Expectation Variance

- $E[X] = \sum x \; P[X=x]$ Expectation
- $Var[X] = E[X\ ] - E[X]$ Variance
- $\sigma[X] = \sqrt{Var[X]}$ Standard deviation
- $Cov[X,Y] = E[X\,Y] - E[X]\,E[Y]$ Covariance
- $Cor[X,Y]=Cov[X,Y]/(\sigma[X]\,\sigma[Y])$ Correlation

Pythagoras

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X,Y]$$

Example

throw a dice

$X(k) = 2$ if prime, else 0

$Y(k) = 3$ if k^2+1, else 0

Find, E[X],E[Y],Var[X],Var[Y], Cov[X,Y]

Part II: Linear Algebra

- Matrices and Matrix Algebra
- Solving Systems of Linear Equations
- Linear Transformations
- Basis of image and kernel
- Change of Coordinates, Similarity

1. Matrices

n x m matrix

n=4

$$\begin{vmatrix} 3 & 1 & 4 & 5 & 1 \\ 1 & 1 & 1 & 2 & 2 \\ 2 & 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 5 & -3 \end{vmatrix}$$

m=5

n x m matrix

n=4

3	1	4	5	1
1	1	1	2	2
2	1	4	1	1
1	1	4	5	-3

m=5

n x m matrix

n=4

$$\begin{vmatrix} 3 & 1 & 4 & 5 & 1 \\ 1 & 1 & 1 & 2 & 2 \\ 2 & 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 5 & -3 \end{vmatrix}$$

m=5

n x m matrix

n=4

3	1	4	5	1
1	1	1	2	2
2	1	4	1	1
1	1	4	5	-3

m=5

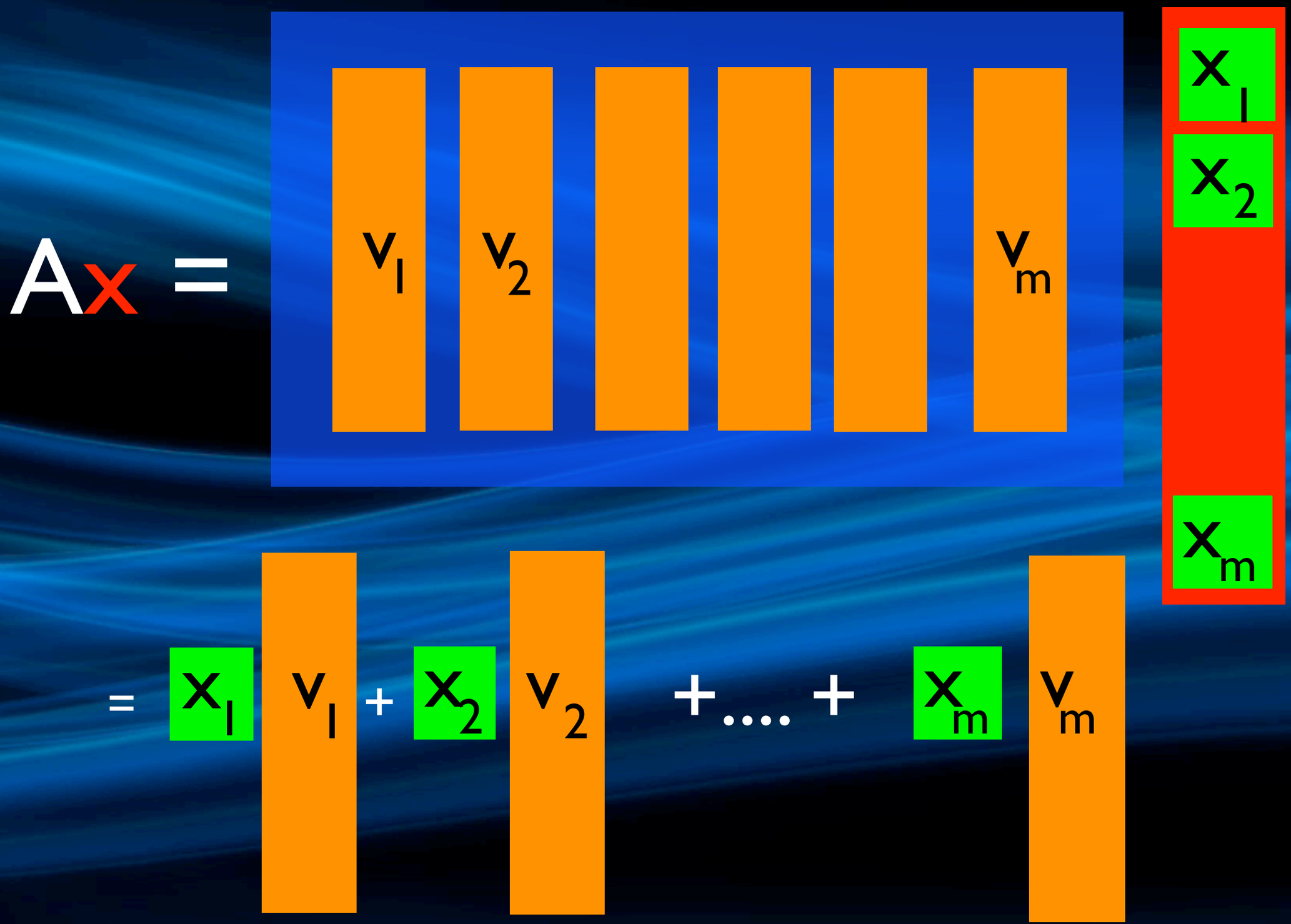
n x m matrix

n=4

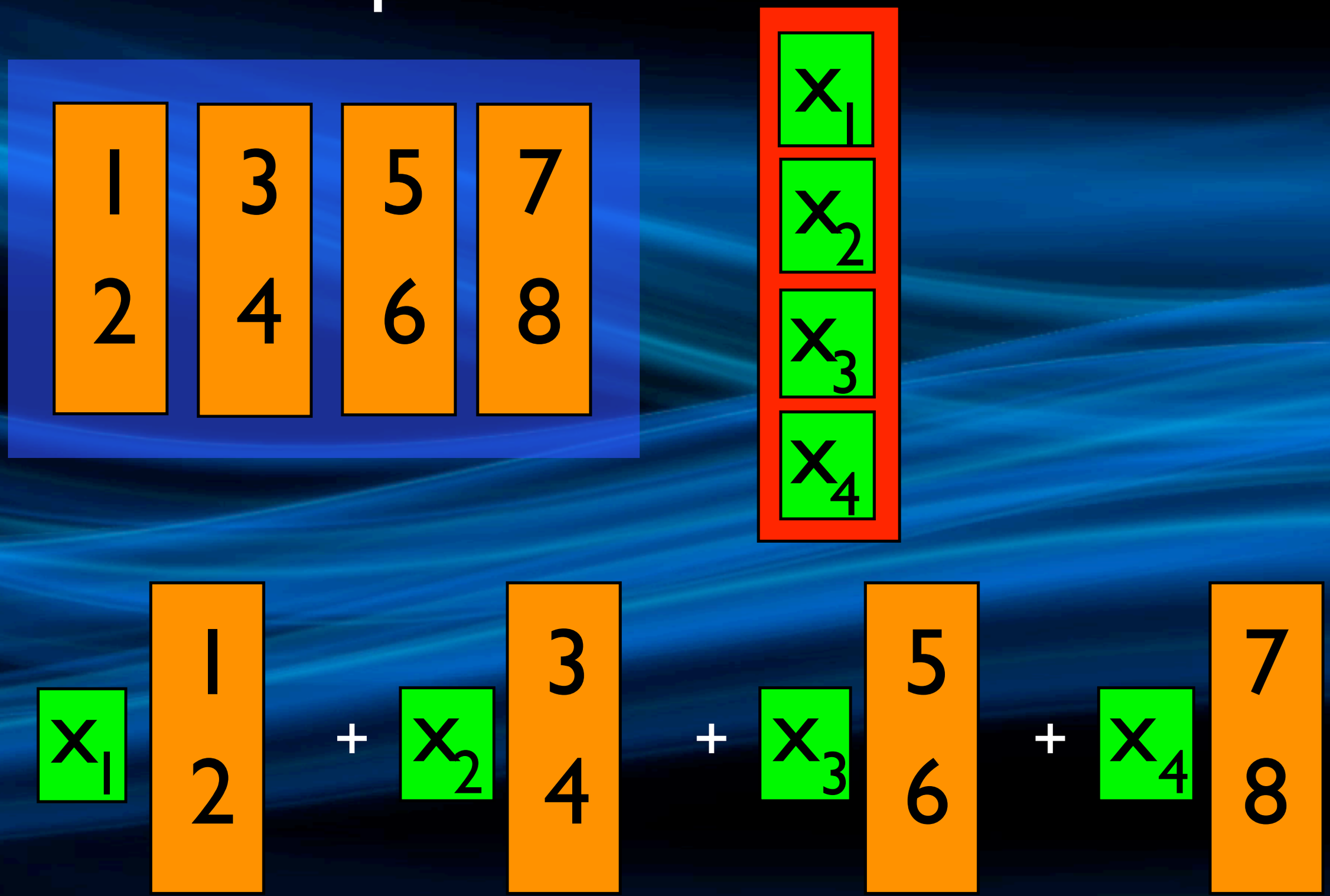
$$\begin{vmatrix} 3 & 1 & 4 & 5 & 1 \\ 1 & 1 & 1 & 2 & 2 \\ 2 & 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 5 & -3 \end{vmatrix}$$

m=5

Column picture



Example



Row picture

$Ax=$

$$Ax = 0$$

$$Ax = b$$

Example

to find all vectors perpendicular to these two vectors:

$$\begin{bmatrix} 1 \\ 2 \\ 4 \\ 6 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 4 \\ 6 \\ 8 \end{bmatrix}$$

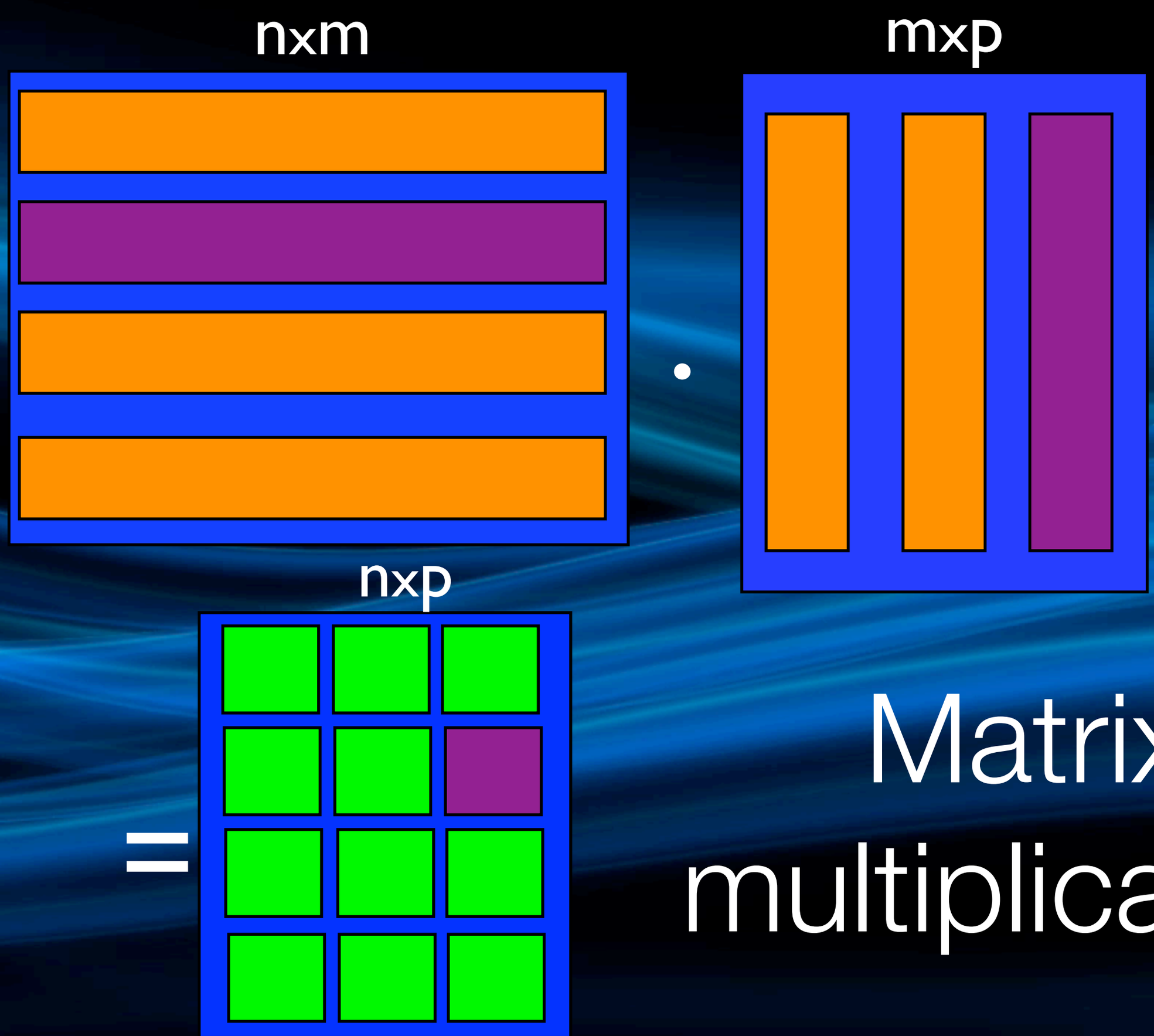
Example

to find all vectors perpendicular to these two vectors:

$$\begin{matrix} 1 \\ 2 \\ 4 \\ 6 \end{matrix} \qquad \begin{matrix} 1 \\ 4 \\ 6 \\ 8 \end{matrix}$$

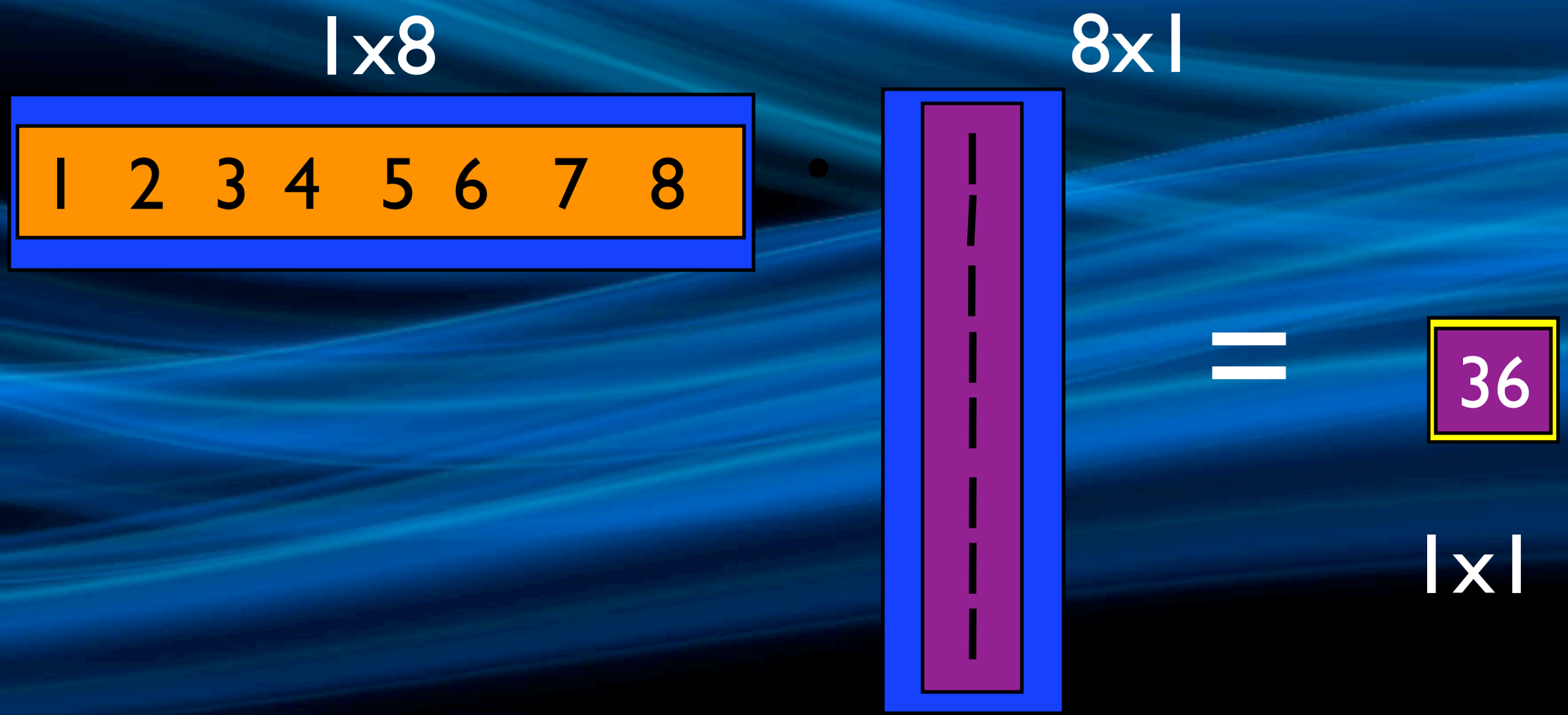
$$A\,x = \begin{bmatrix} 1 & 2 & 4 & 6 \\ 1 & 4 & 6 & 8 \end{bmatrix} \begin{matrix} x \end{matrix}$$

solve $A\,x = 0$, where A contains the vectors as rows.

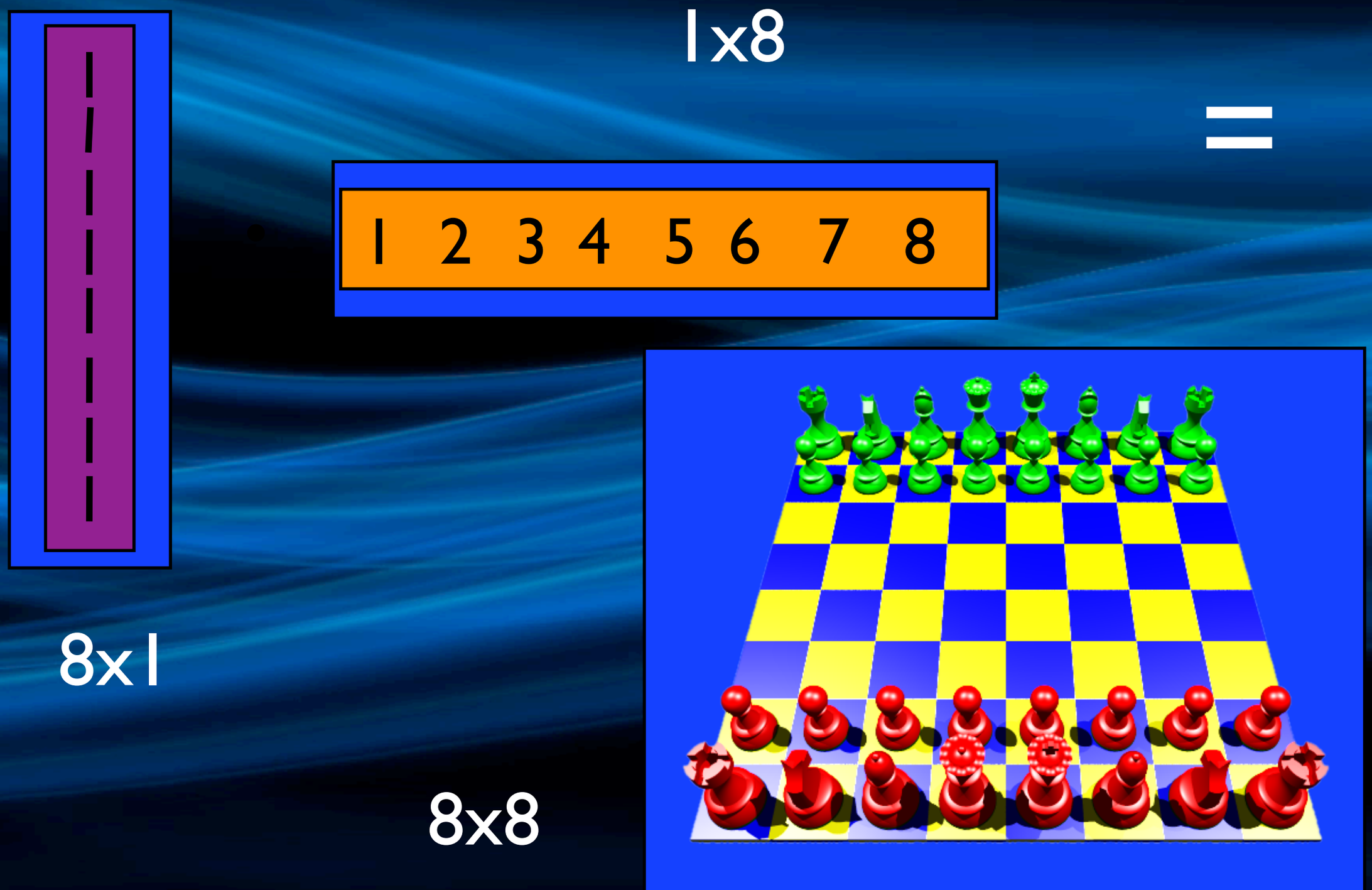


Matrix multiplication

Example



Example



Matrix Algebra

$$\underset{n \times n}{\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}} \bullet \underset{n \times n}{\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}}$$

$$= \underset{n \times n}{\begin{bmatrix} -1 & -2 & 0 \\ 2 & 1 & 2 \\ -1 & -2 & 0 \end{bmatrix}}$$

Not Commutative

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

rotate reflect

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

reflect rotate

Not invertible in general

$$\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & I \\ I & I \end{bmatrix}$$

Blackboard Problems:

True or False:

- $(A-B)(A+B) = A^2 - B^2$
- $B = S^{-1} A S$ implies $A = S B S^{-1}$
- $A X B = A - B$ implies $X = B^{-1} - A^{-1}$
- A and $B = S^{-1} A S$ are similar

2. Linear Equations

Linear Equations

$$2x + y + 5z + w = 3$$
$$2x + 2z - 2w = 6$$

Linear Equations

$$2x + y + 5z + w = 3$$
$$2x + 2z - 2w = 6$$

$$\begin{bmatrix} 2 & 1 & 5 & 1 \\ 2 & 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Linear Equations

$$\begin{bmatrix} 2 & 1 & 5 & 1 \\ 2 & 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

Linear Equations

$$\begin{bmatrix} 2 & 1 & 5 & 1 \\ 2 & 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 1 & 5 & 1 & 3 \\ 2 & 0 & 2 & -2 & 6 \end{array}\right]$$

row
reduction

2	1	5	1	3
2	0	2	-2	6

row reduction

$$\left[\begin{array}{cccc|c} 2 & 1 & 5 & 1 & 3 \\ 2 & 0 & 2 & -2 & 6 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 3 \\ 2 & 1 & 5 & 1 & 3 \end{array}\right]$$

row reduction

$$\left[\begin{array}{cccc|c} 2 & 1 & 5 & 1 & 3 \\ 2 & 0 & 2 & -2 & 6 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 3 \\ 2 & 1 & 5 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 3 \\ 0 & 1 & 3 & 5 & -3 \end{array}\right]$$

row reduction

1	0	1	-1	3
0	1	3	5	-3
		s	t	

1	0	1	-1	3
0	1	3	5	-3
		s	t	

$$x + s - t = 3$$

$$y+3s+5t=-3$$

$$z=s$$

$$w=t$$

		s	t	
1	0	1	-1	3
0	1	3	5	-3

$$x + s - t = 3$$

$$y+3s+5t=-3$$

$$z=s$$

$$w=t$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = s \begin{bmatrix} -1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 5 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -3 \\ 0 \\ 0 \end{bmatrix}$$

Gauss-Jordan Elimination

Scale a row

S

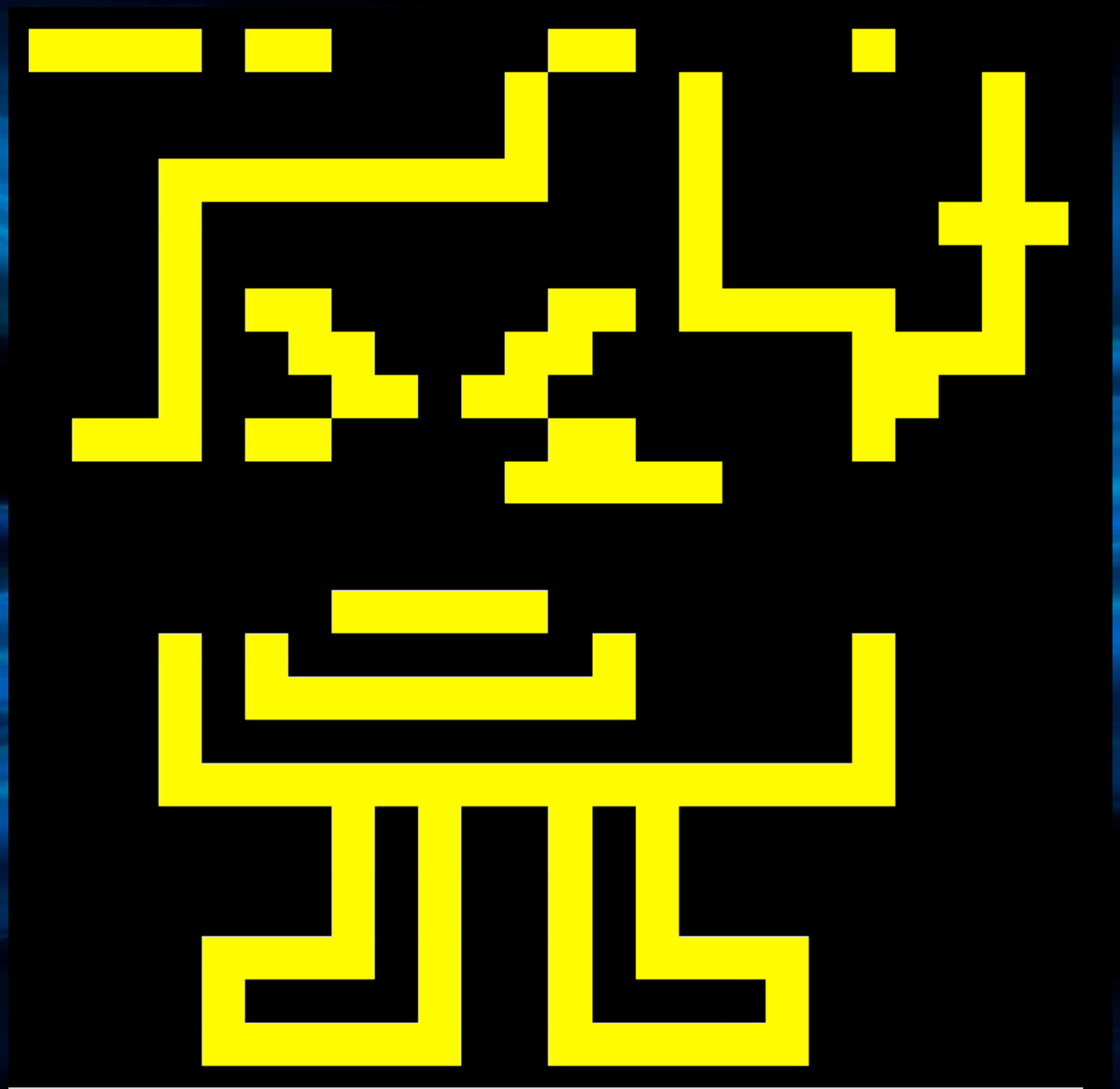
Swap two rows

S

Subtract row from other row

S

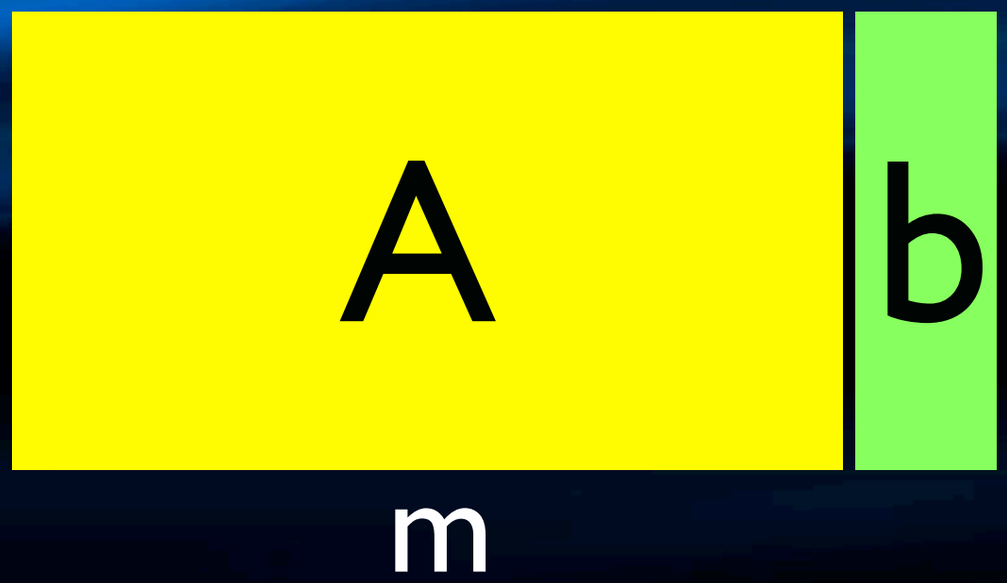
rref death of milkshake



Number of Solutions

Number of Solutions

$B=[A|b] =$



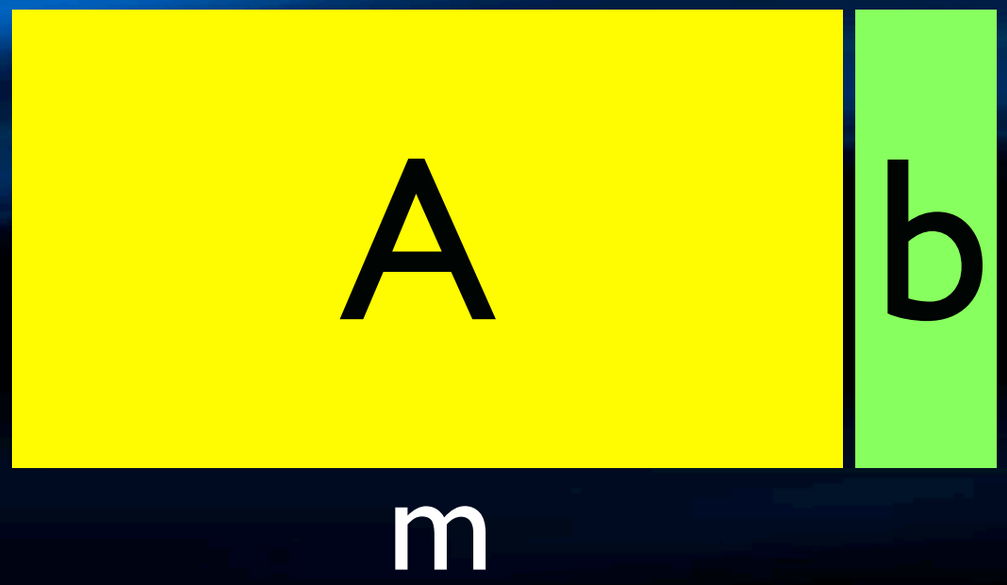
Number of Solutions

one solution

consistent

rank(A)=m

$B=[A|b] =$



Number of Solutions

one solution

consistent

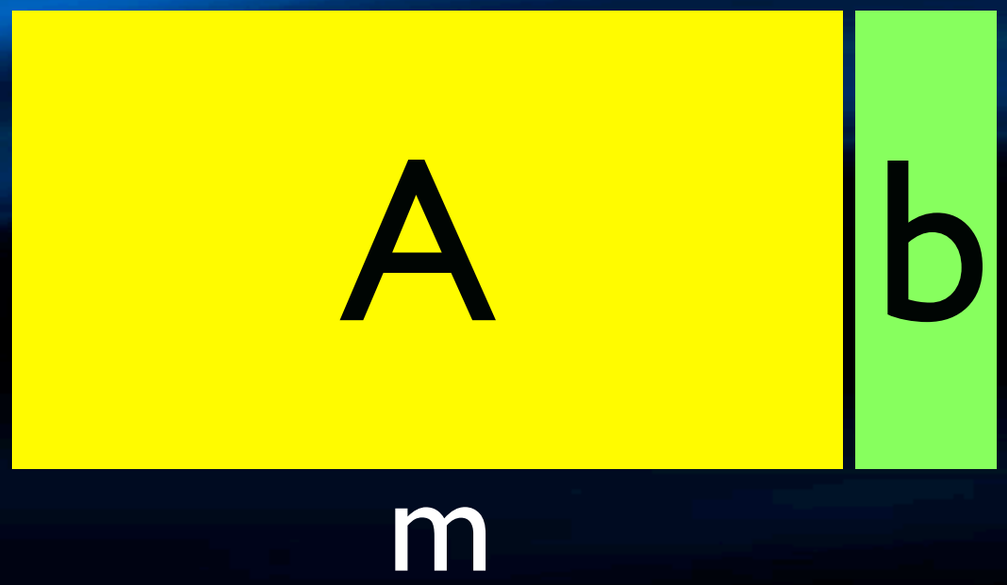
rank(A)=m

zero solutions

inconsistent

rank(B)>rank(A)

$B=[A|b] =$



Number of Solutions

one solution

consistent

rank(A)=m

zero solutions

inconsistent

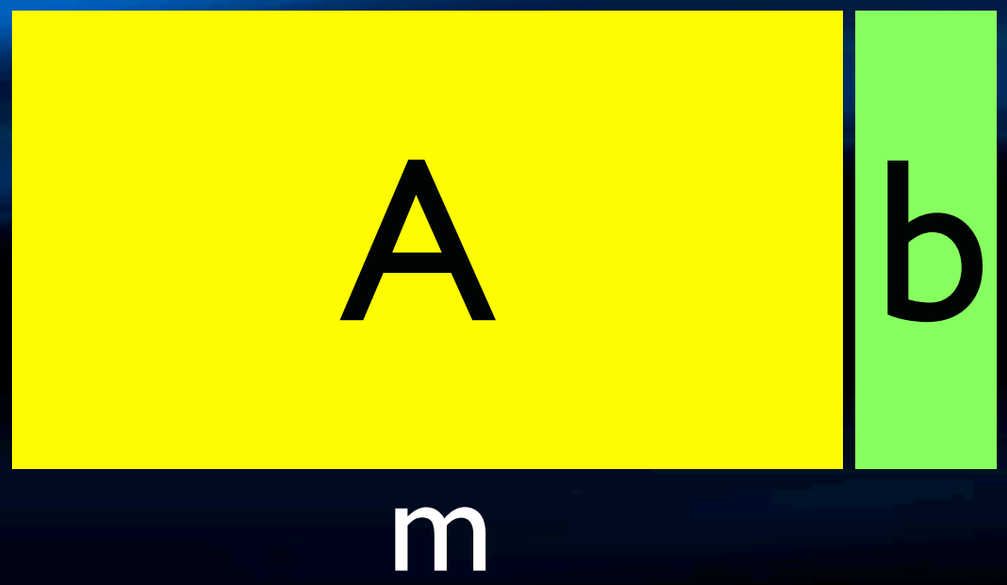
rank(B)>rank(A)

infinitely many

consistent

rank(A)=rank(B) < m

$B=[A|b] =$



How many solutions?

augmented matrix

$$\left[\begin{array}{cccccccc|c}
1 & 1 & 1 & 6 & 0 & 0 & 1 & 2 \\
0 & 0 & 1 & 2 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

Row reduced properties

0	1	4	0
0	0	0	1

Row reduced properties

First nonzero row element is 1

0	1	4	0
0	0	0	1

Row reduced properties

First nonzero row element is 1

Column with leading 1 has no other nonzero entry

0	1	4	0
0	0	0	1

Row reduced properties

First nonzero row element is 1

Column with leading 1 has no other nonzero entry

Row above row with leading 1 has leading 1 to the left

$$\begin{matrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

row reduced?

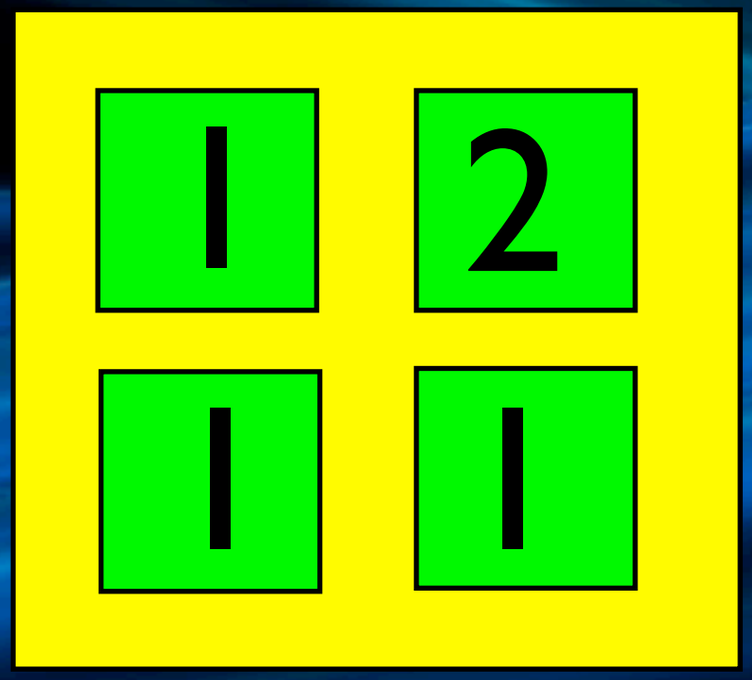
1	0	8	8	8
0	1	8	8	8
0	0	0	0	0
0	0	0	0	0

row reduced?

1	8	0	0	0	8
0	0	1	0	0	8
0	0	0	1	0	8
0	0	0	0	1	8
0	0	0	0	0	0

invert this matrix

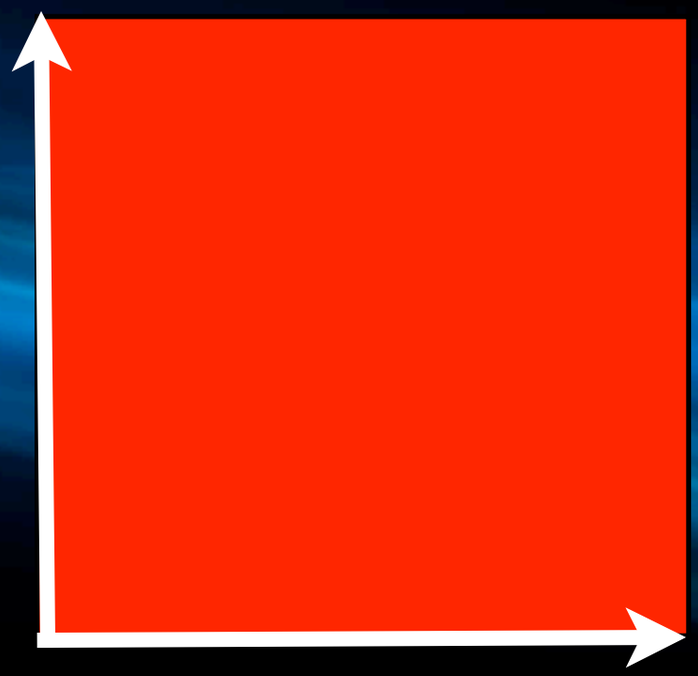
with row reduction

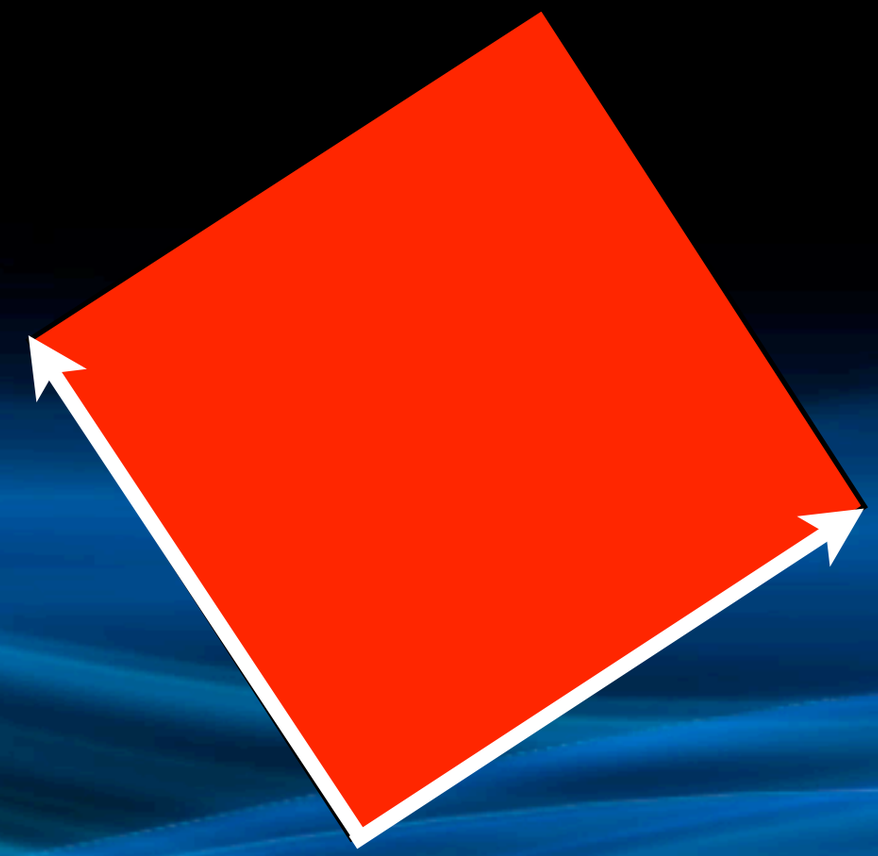


3. Linear Transformations

Look at the image of the basis vectors.

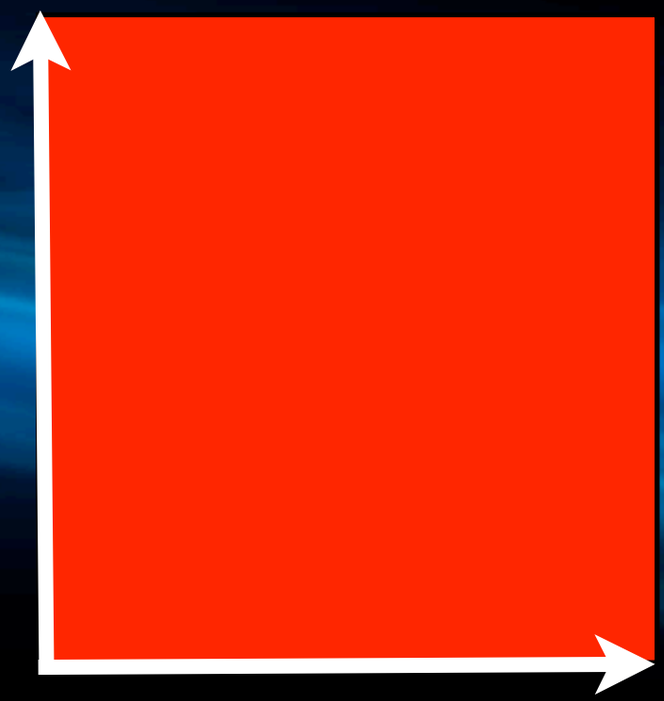
- Rotations
- Reflections
- Projections
- Shears
- Dilations
- Combinations

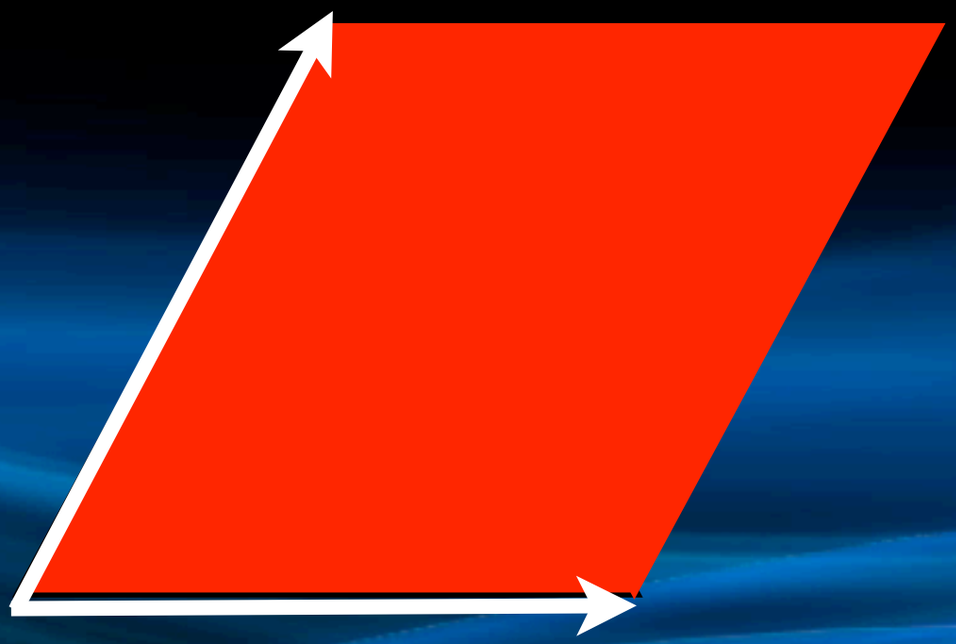




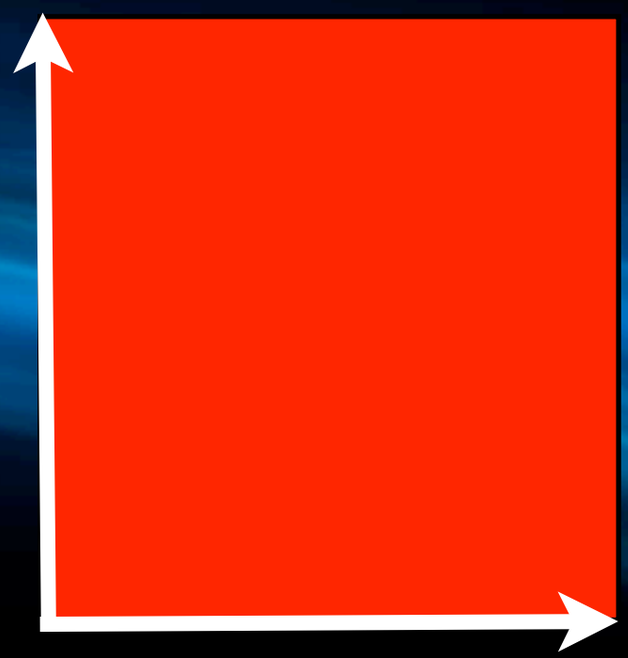
$$\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

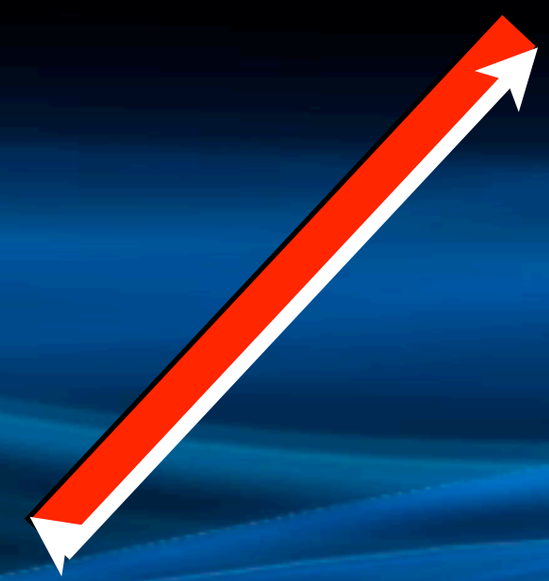
$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$



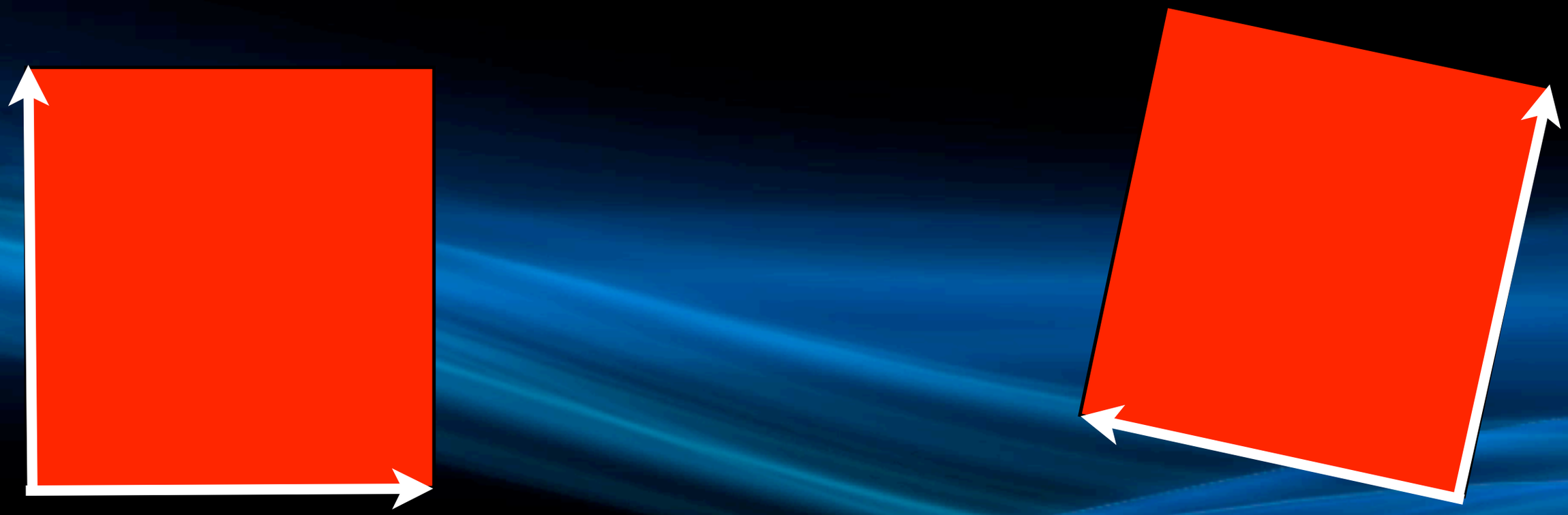


$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

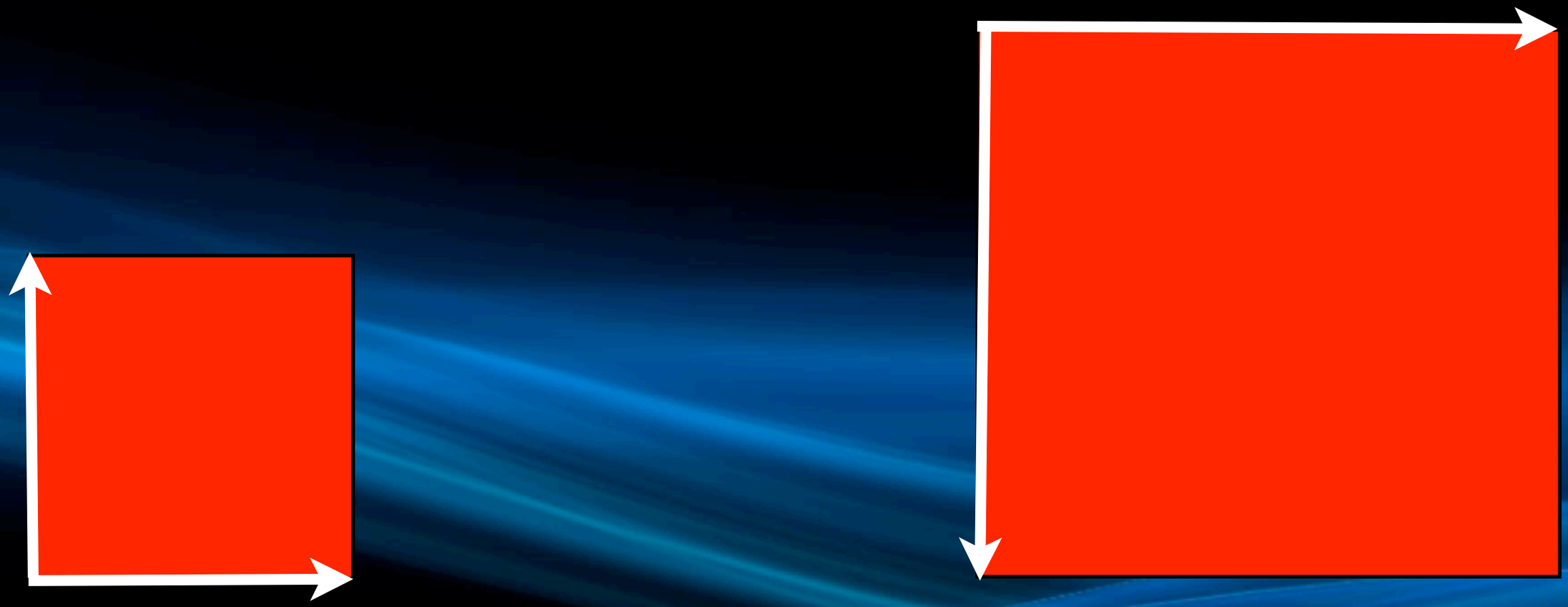




$$\begin{bmatrix} u_1 u_1 & u_2 u_1 \\ u_1 u_2 & u_2 u_2 \end{bmatrix}$$



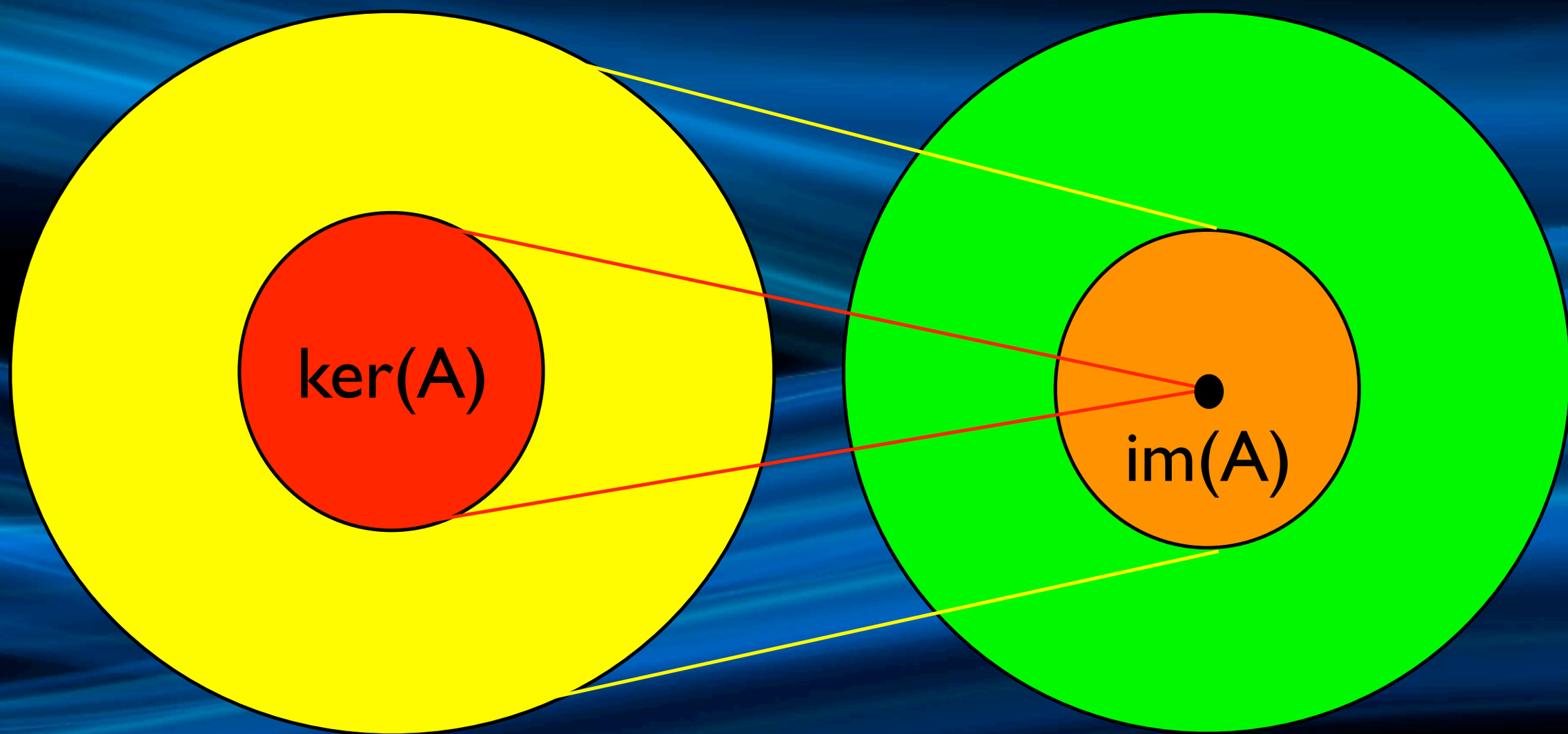
$$\begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$$



$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

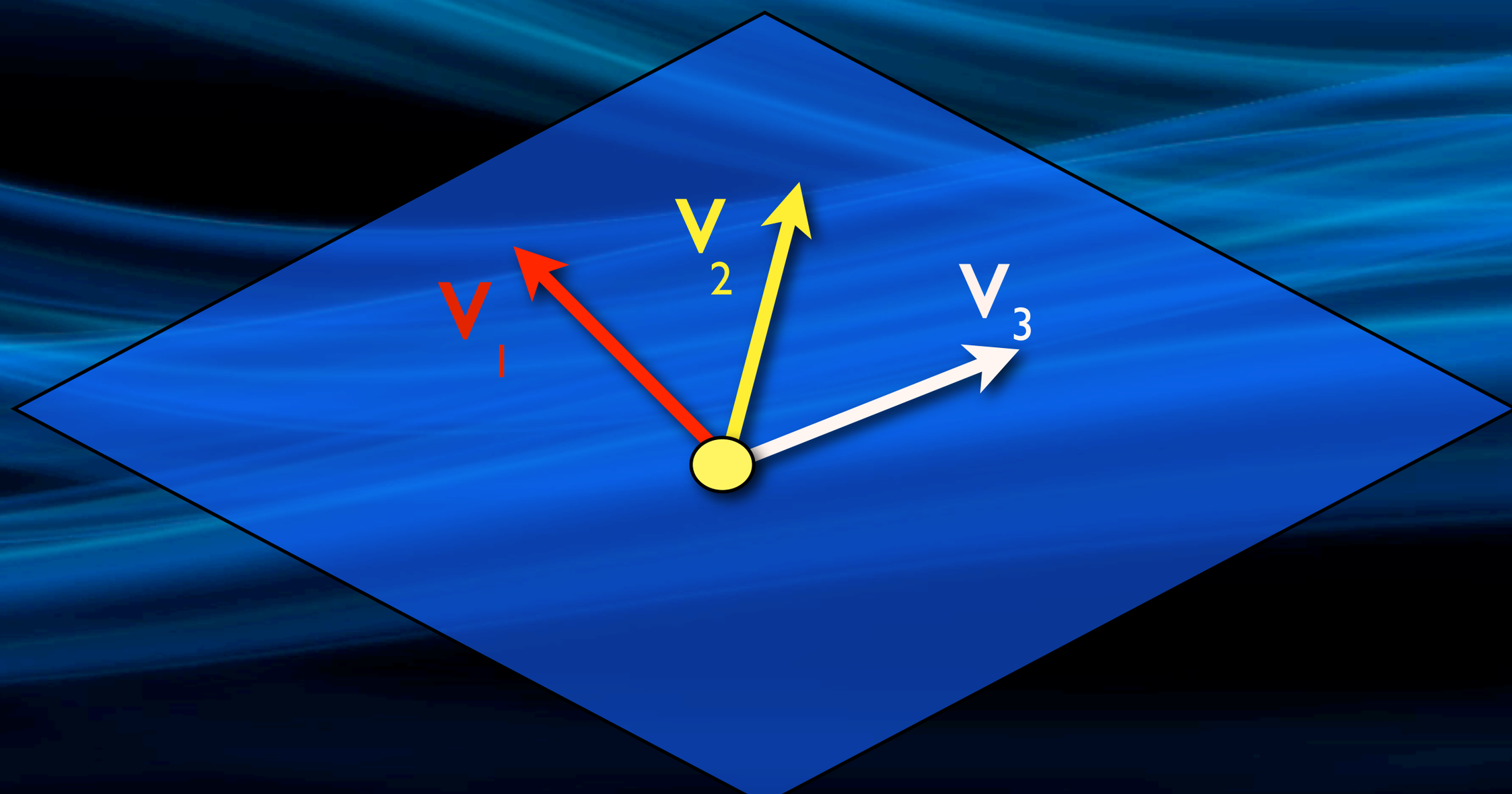
Basis

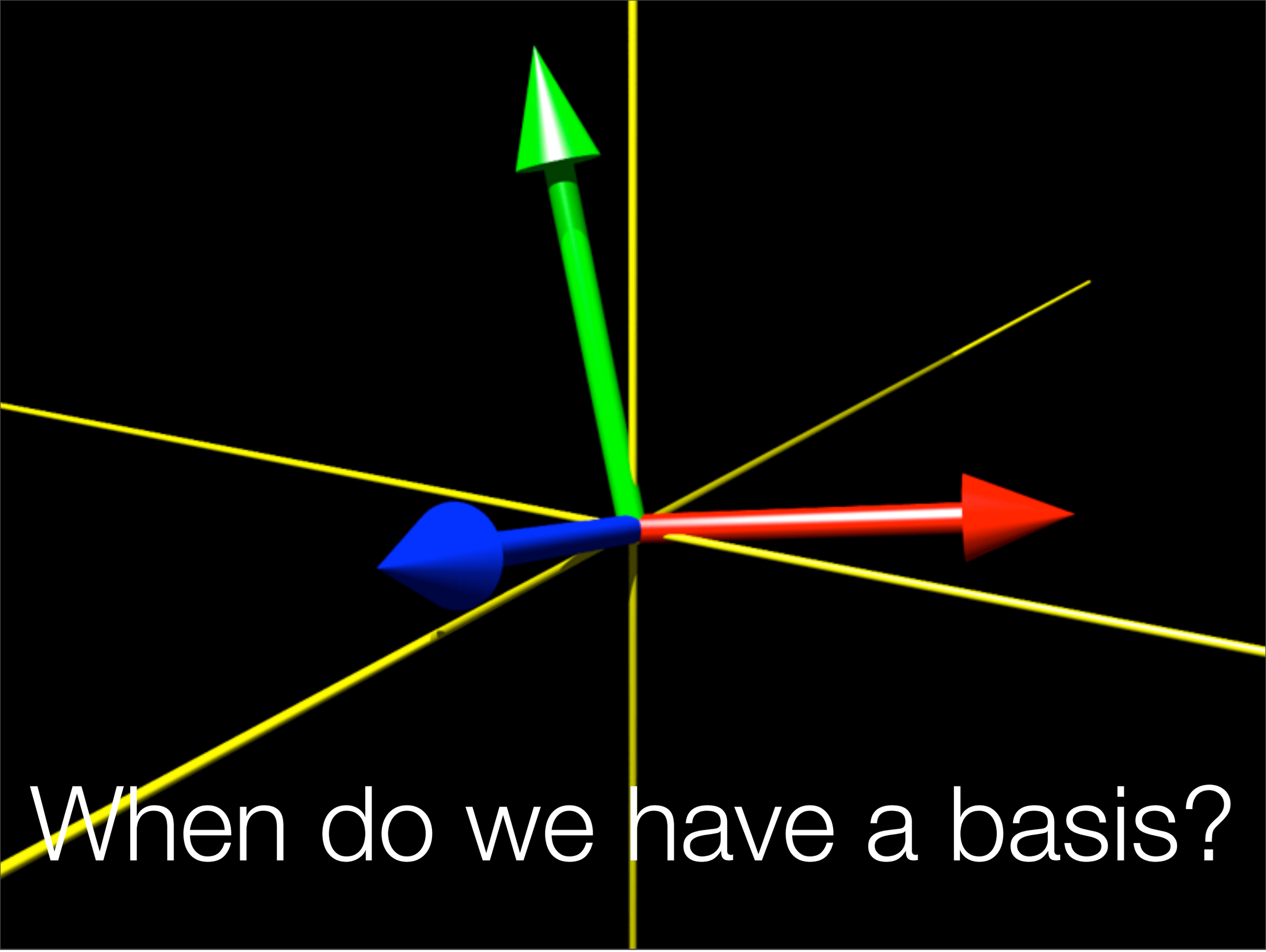
$$\mathrm{im}(A) = \{ Ax \mid x \text{ in } V \}$$



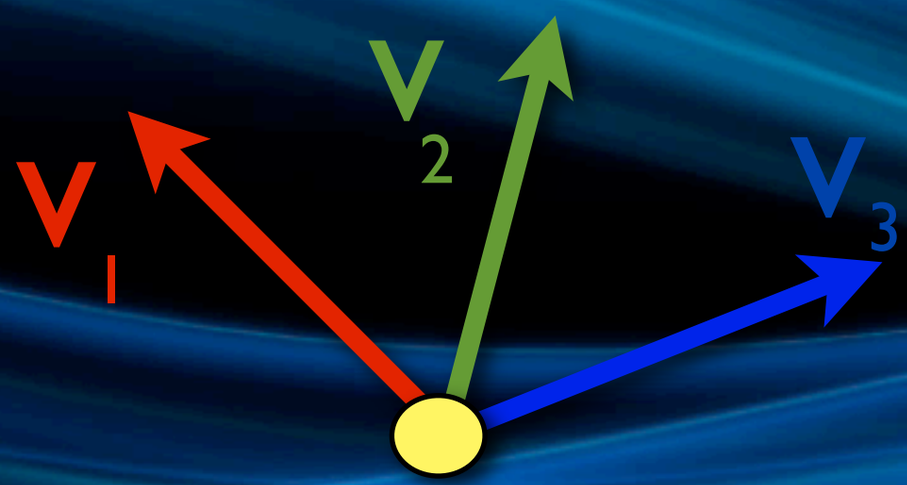
$$\ker(A) = \{ x \mid Ax = 0 \}$$

Basis: linearly independent and spanning

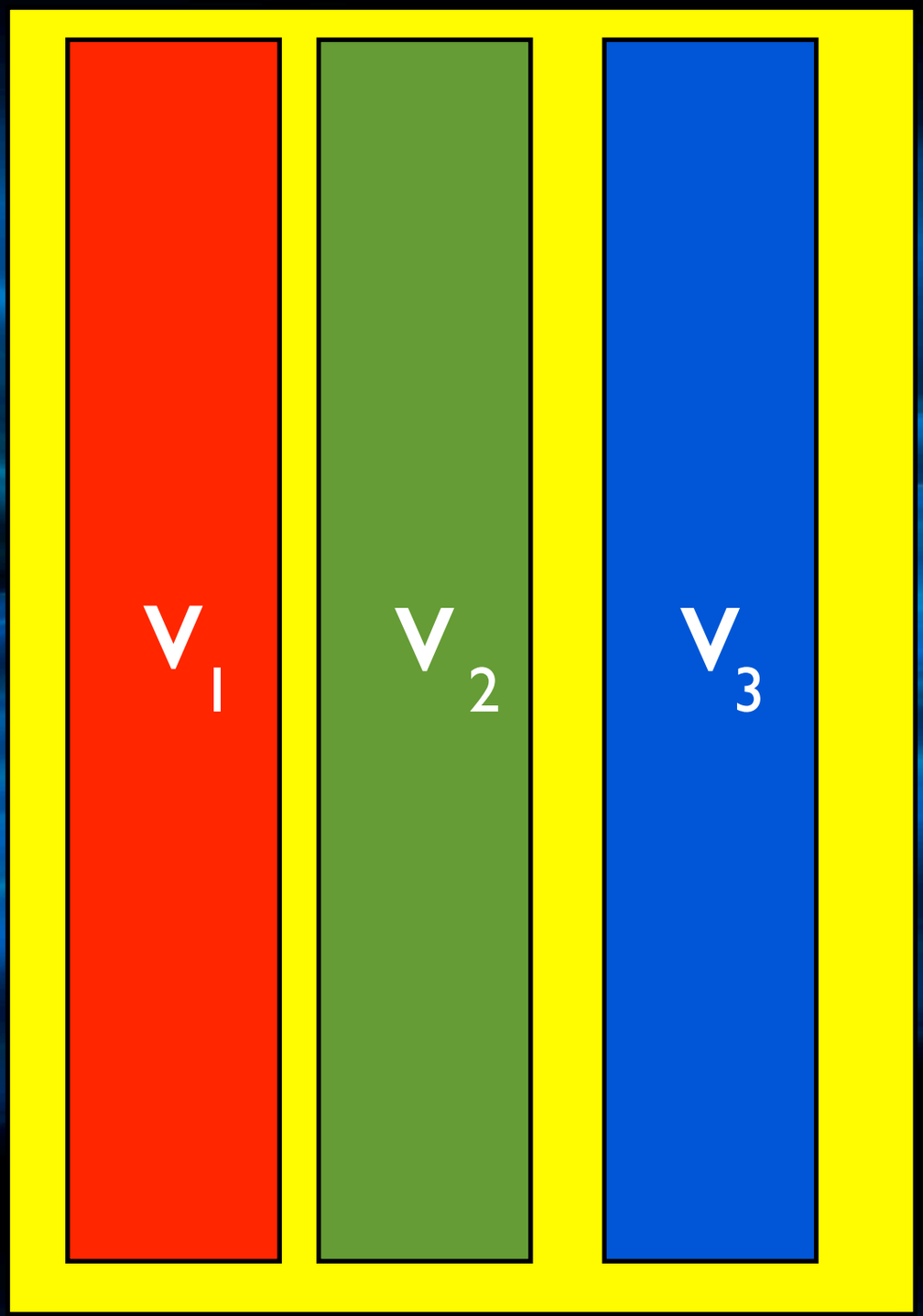



When do we have a basis?

look at the matrix A

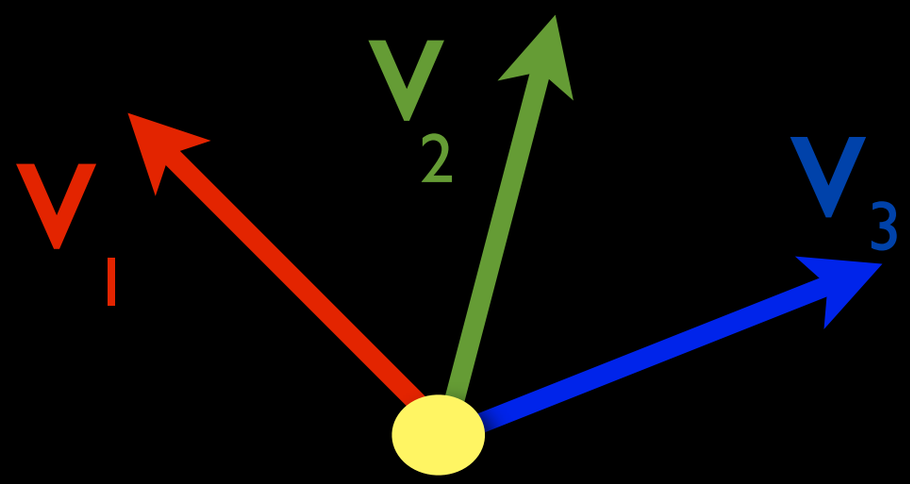


A =

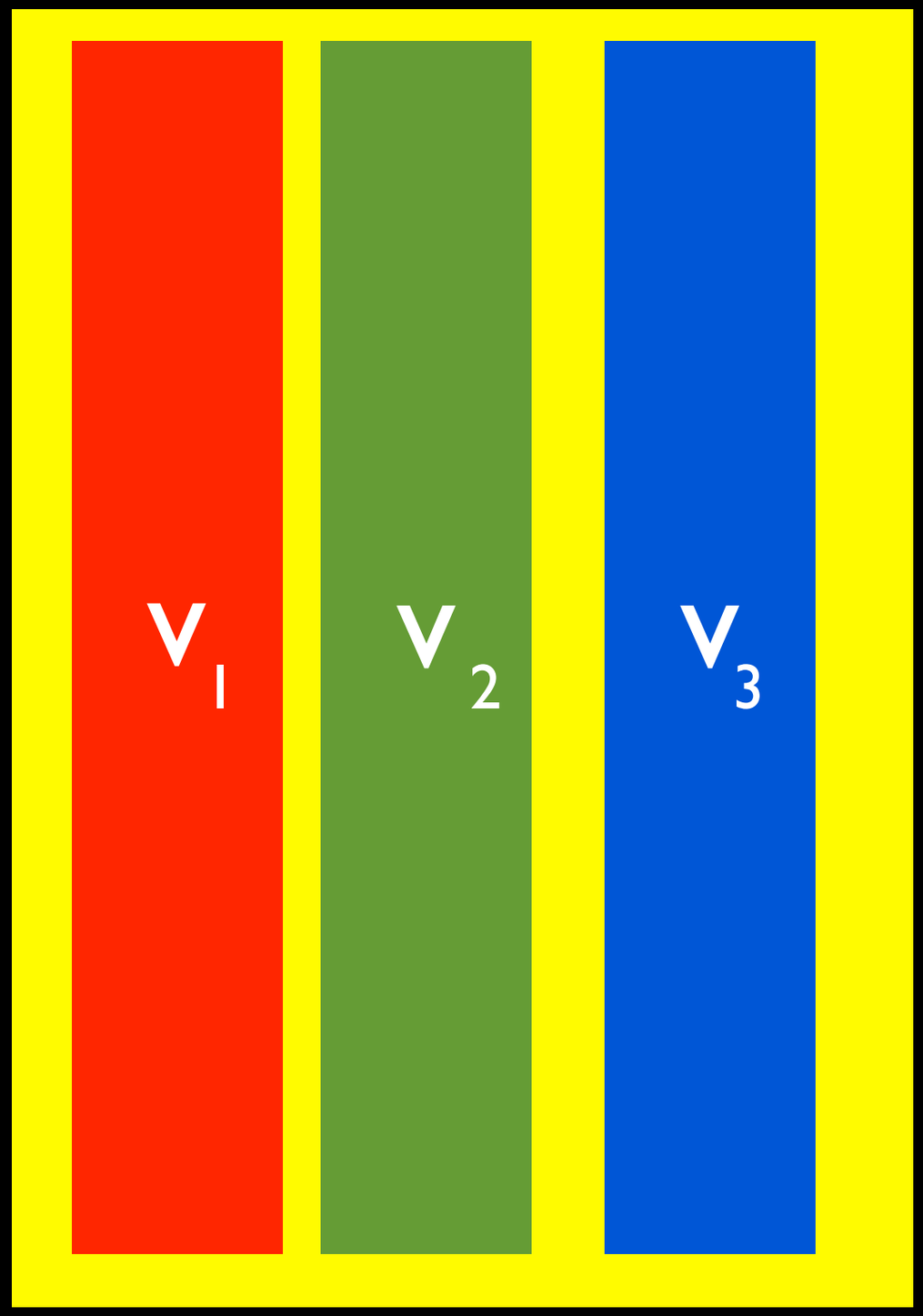


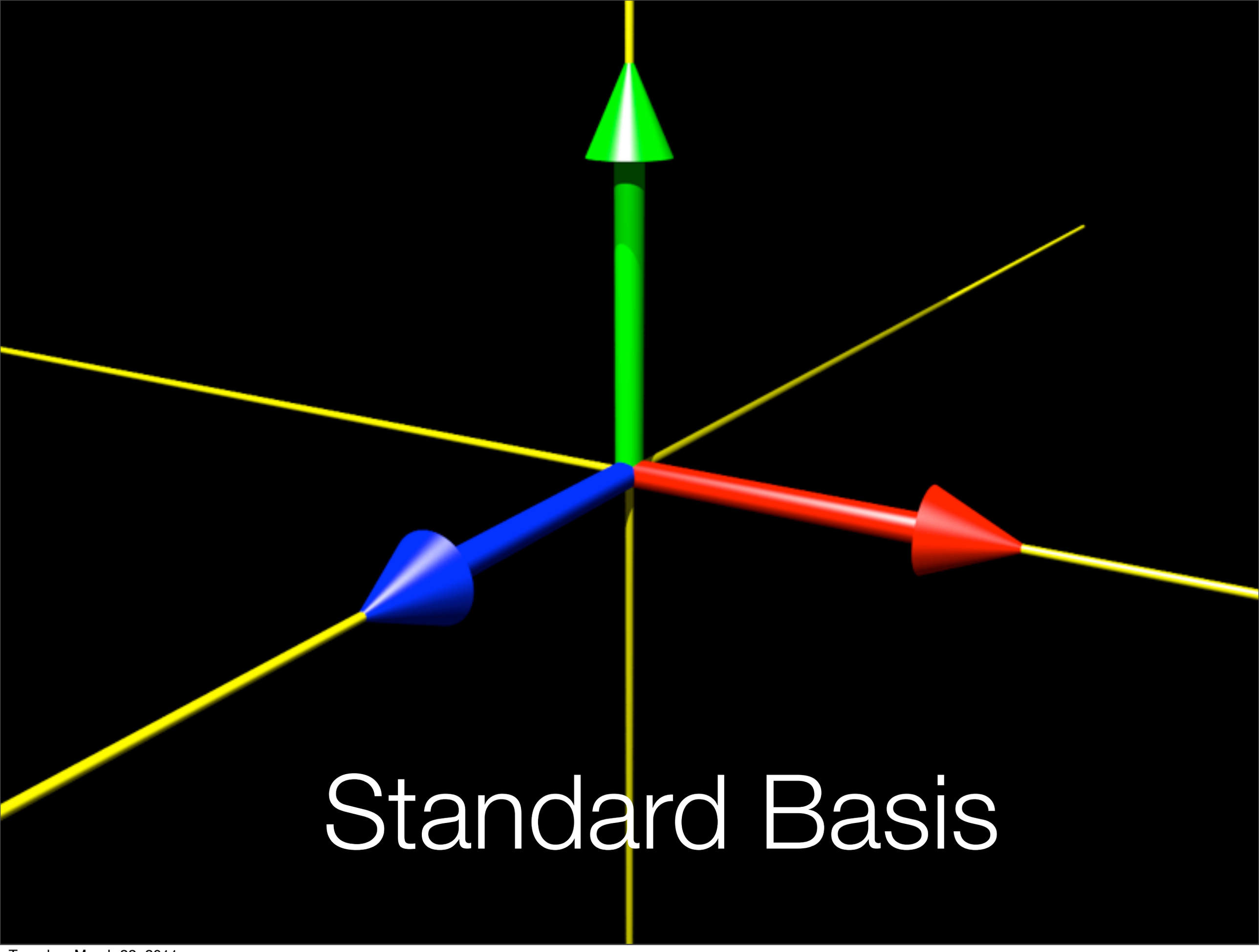
trivial kernel

V is in the image of A

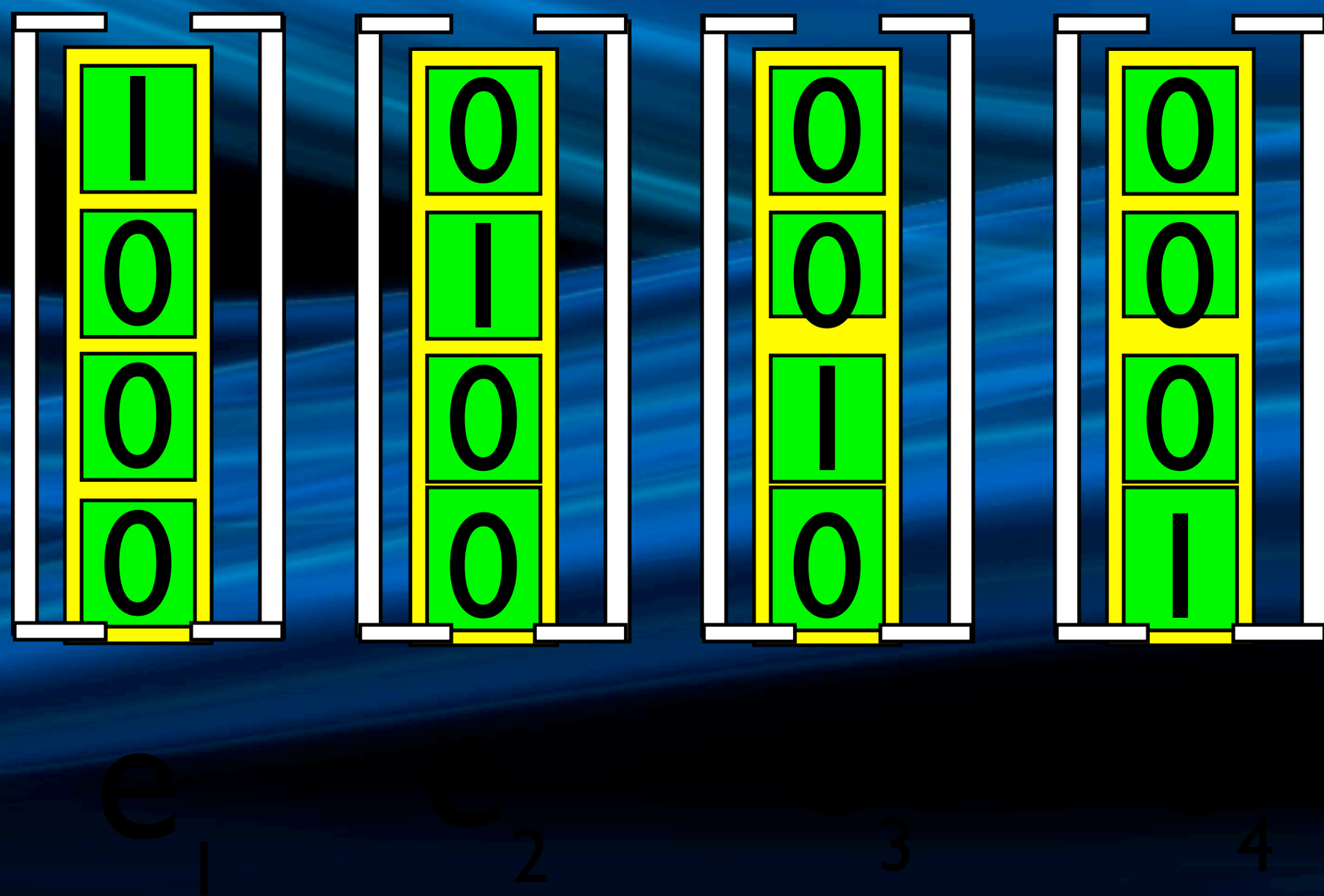


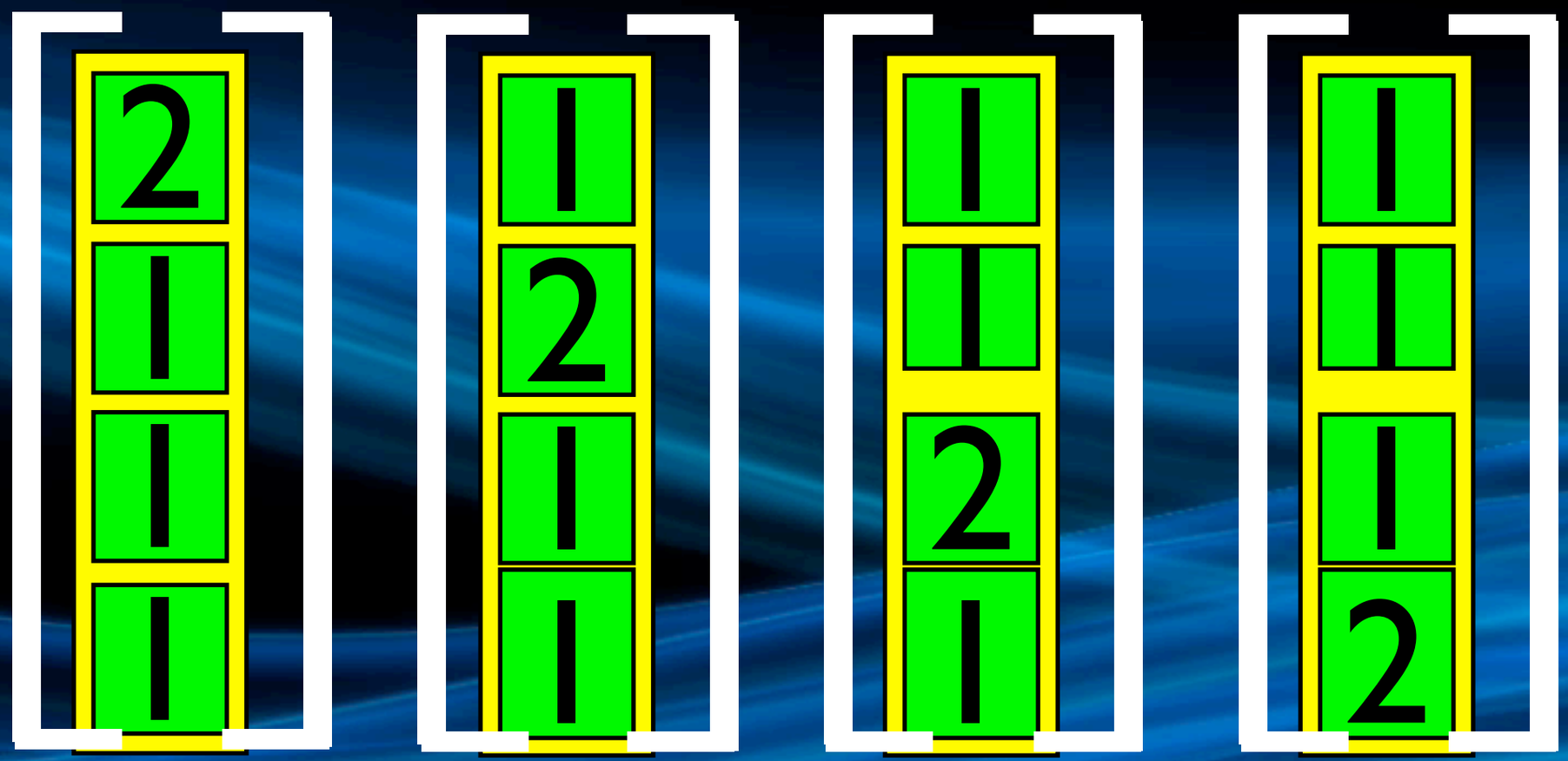
A =




Standard Basis

Standard Basis





v_1 v_2 v_3 v_4

Is this a basis?

Rank-Nullety Theorem

$$\dim(\mathrm{im}(A)) + \dim(\ker(A)) = m$$

Pivot or
not pivot, that is
here the question

n

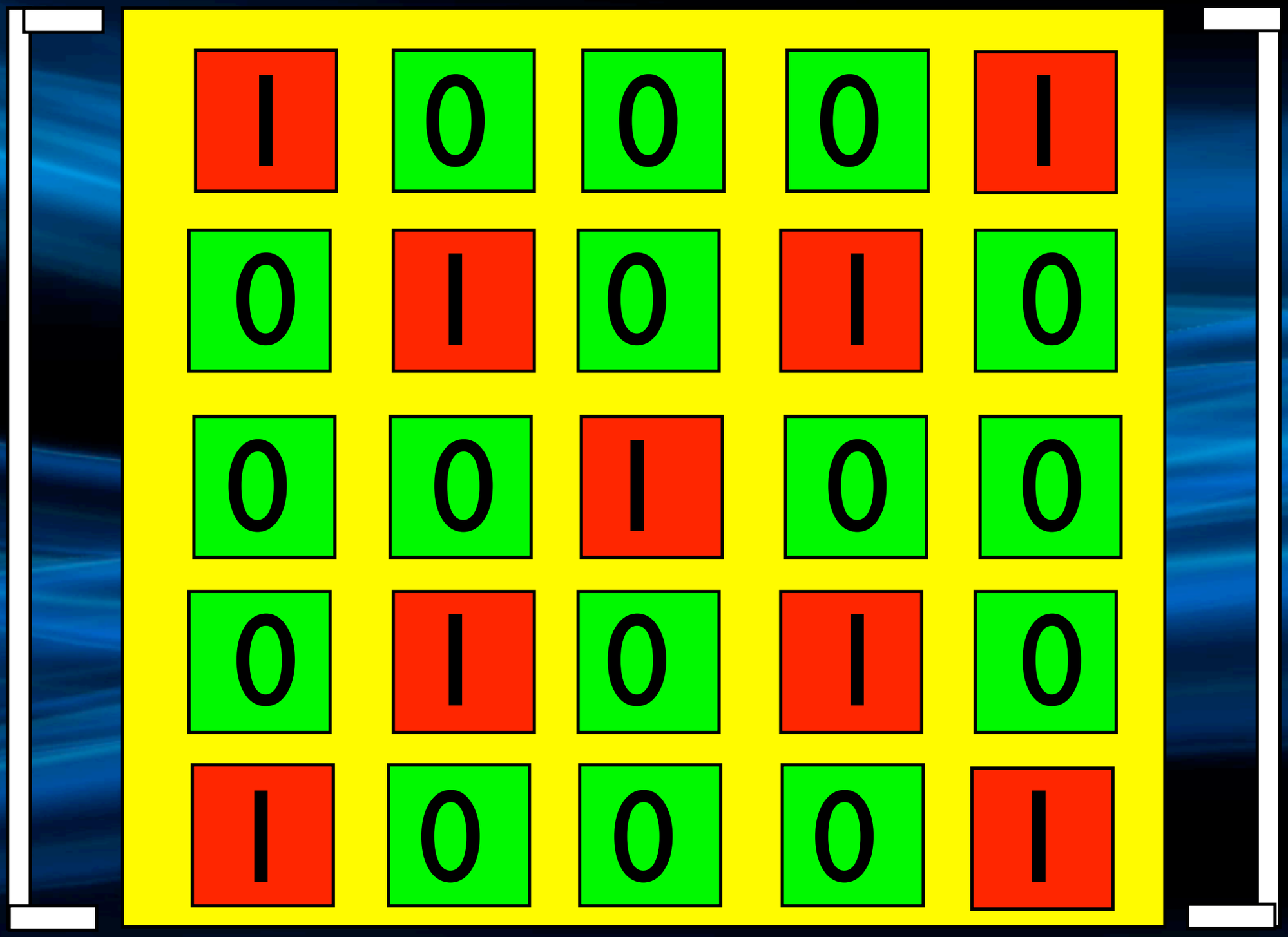
m

Image computation

The basis is the set of pivot columns.

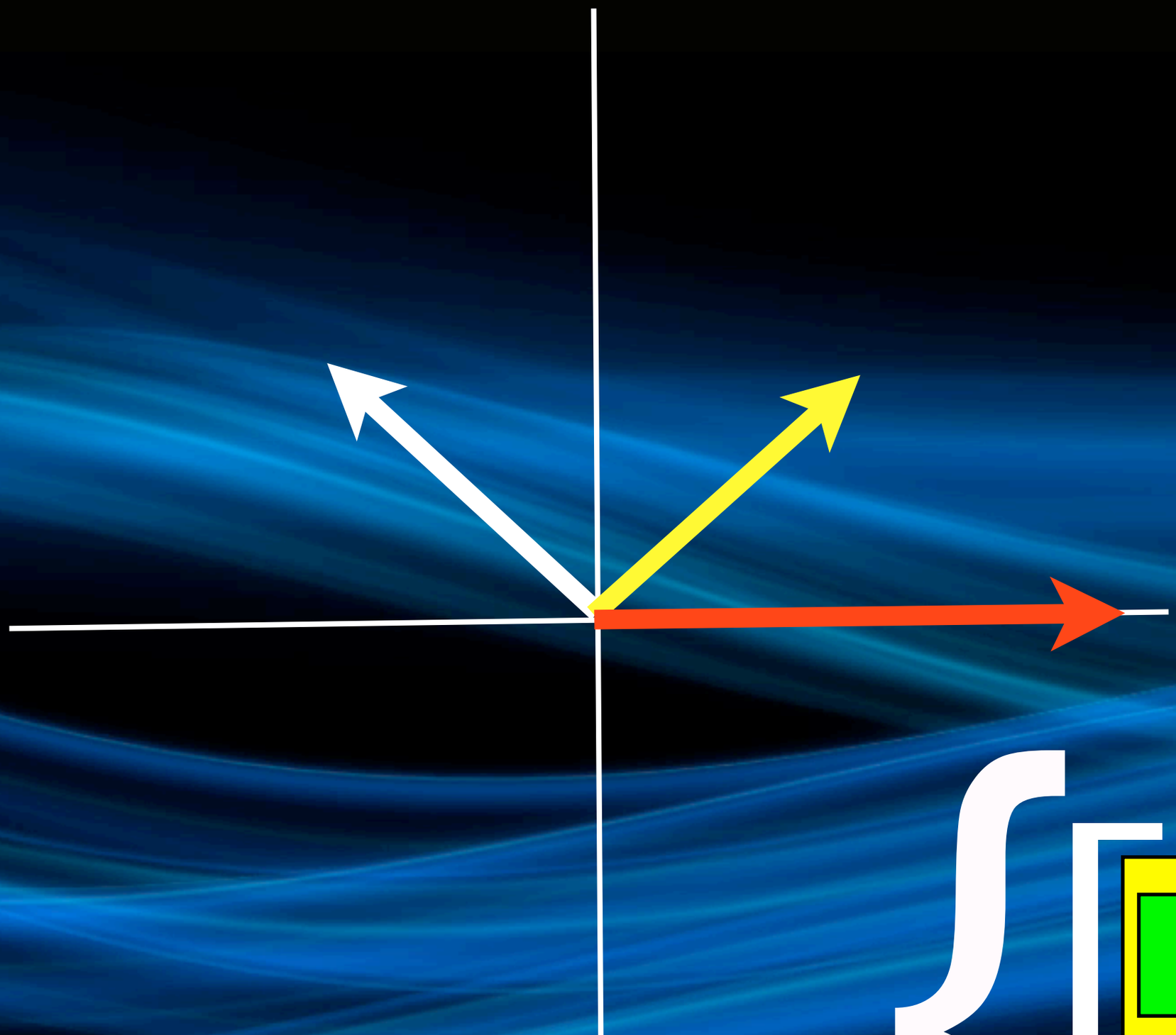
Kernel computation

The basis is obtained by solving the linear system in row reduced echelon form and taking free variables.


1 0 0 0 1
0 1 0 1 0
0 0 1 0 0
0 1 0 1 0
1 0 0 0 1

Coordinates

A vector gets coordinates if a basis is fixed



$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$$

Coordinates in Basis B

$$[v]_B = S^{-1} v$$

$$S = \begin{bmatrix} v_1 & v_2 & & v_n \end{bmatrix}$$

$B = S^{-1} A S$

$$\begin{tikzcd}
\vec{v} \arrow[d, "A"'] & \vec{w} = [\vec{v}]_{\mathcal{B}} \arrow[l, "S"'] \arrow[d, "B"] \\
A\vec{v} \arrow[r, "S^{-1}"] & B\vec{w} = [A\vec{v}]_{\mathcal{B}}
\end{tikzcd}$$

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

What are the B coordinates of v = $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$$S = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S^{-1}v = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}$$

Write A in the basis B

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 6 \end{bmatrix}$$

$$B = \left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

Problem

Find the matrix A of the reflection at the line [-1,1,1]

Similarity

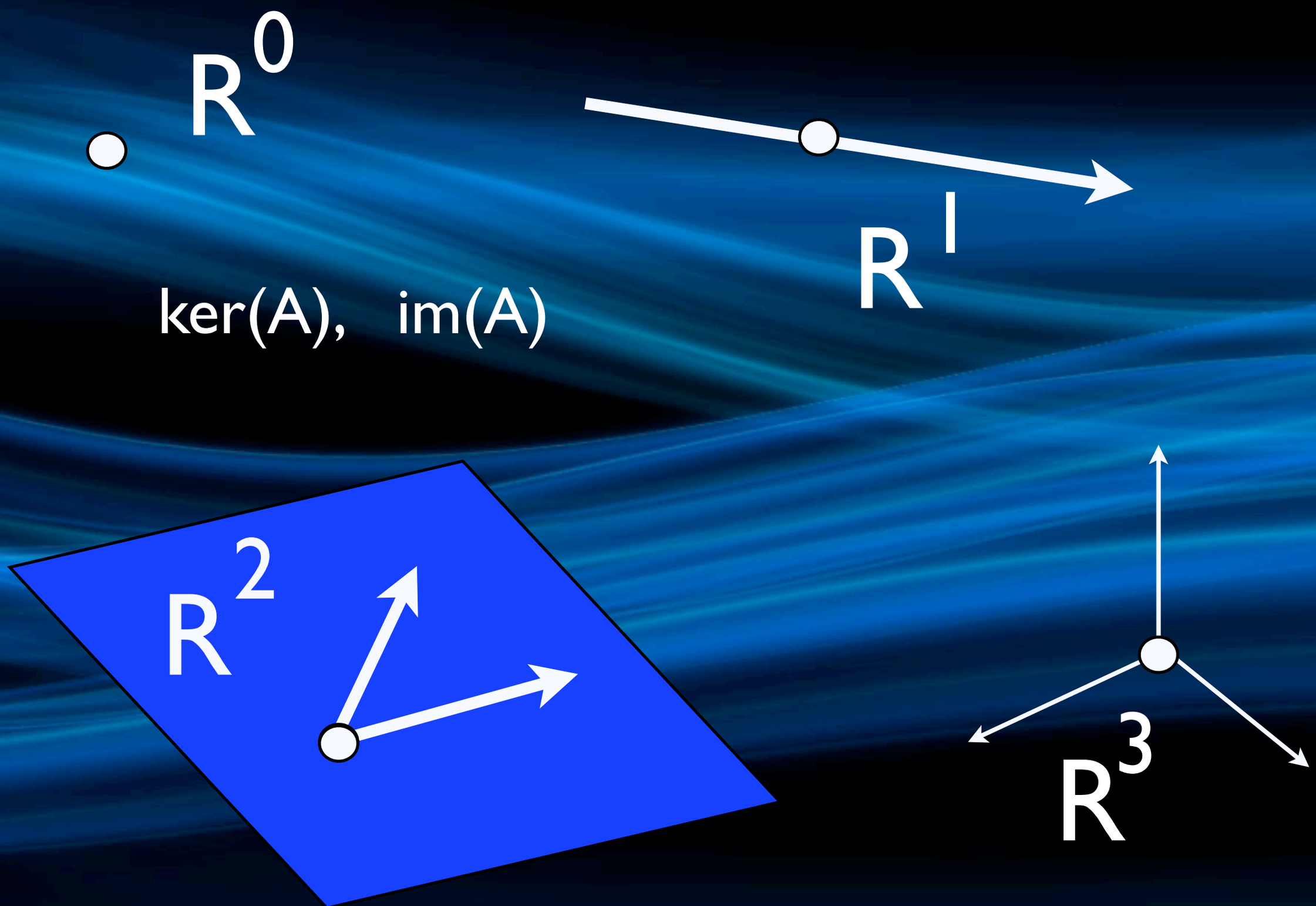
$$B = S^{-1} A S$$

The matrices A,B describe the same transformation!

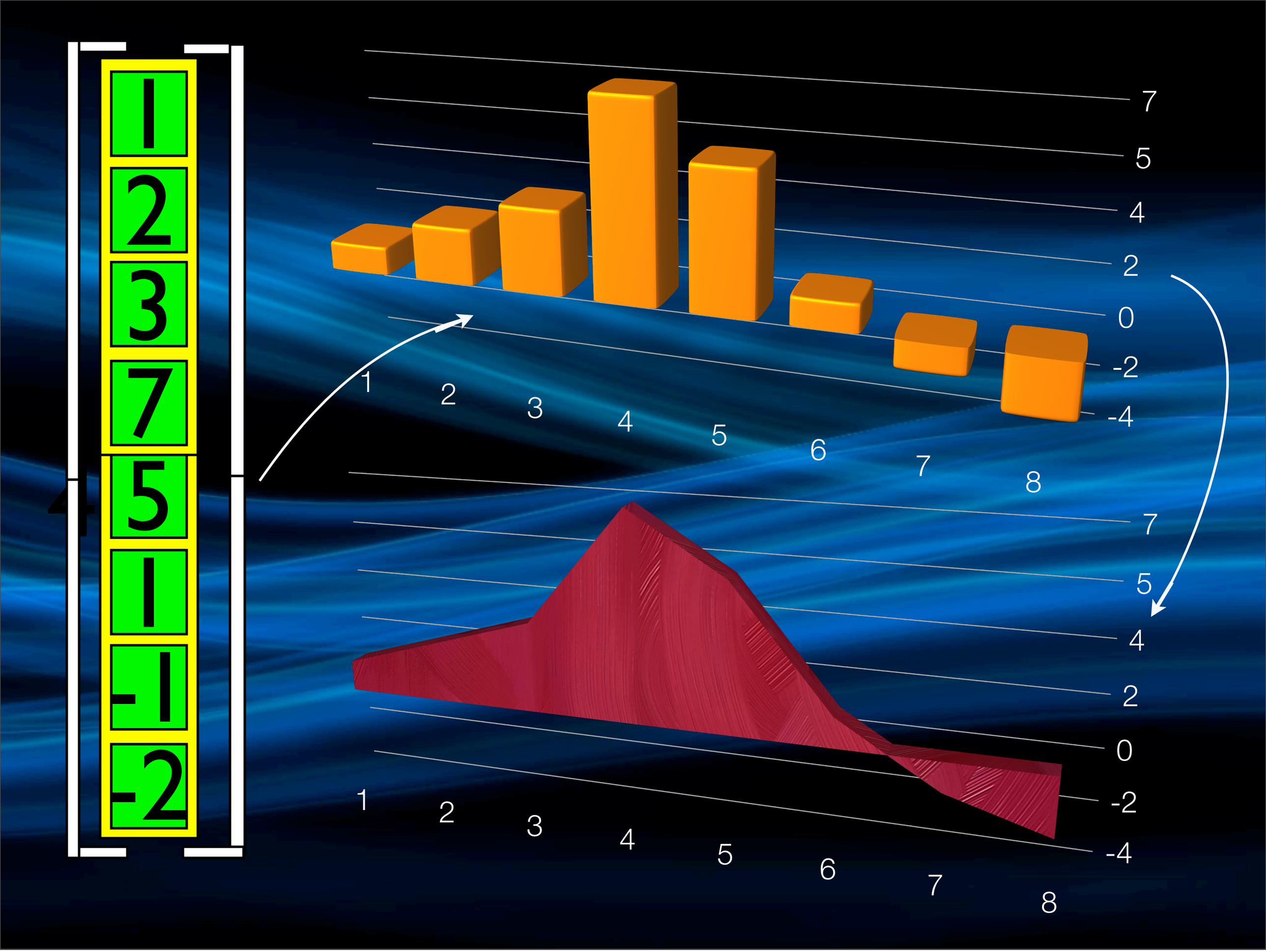
The only matrix similar to the identity matrix is the identity matrix itself.

$$B^n = S^{-1} A^n S$$

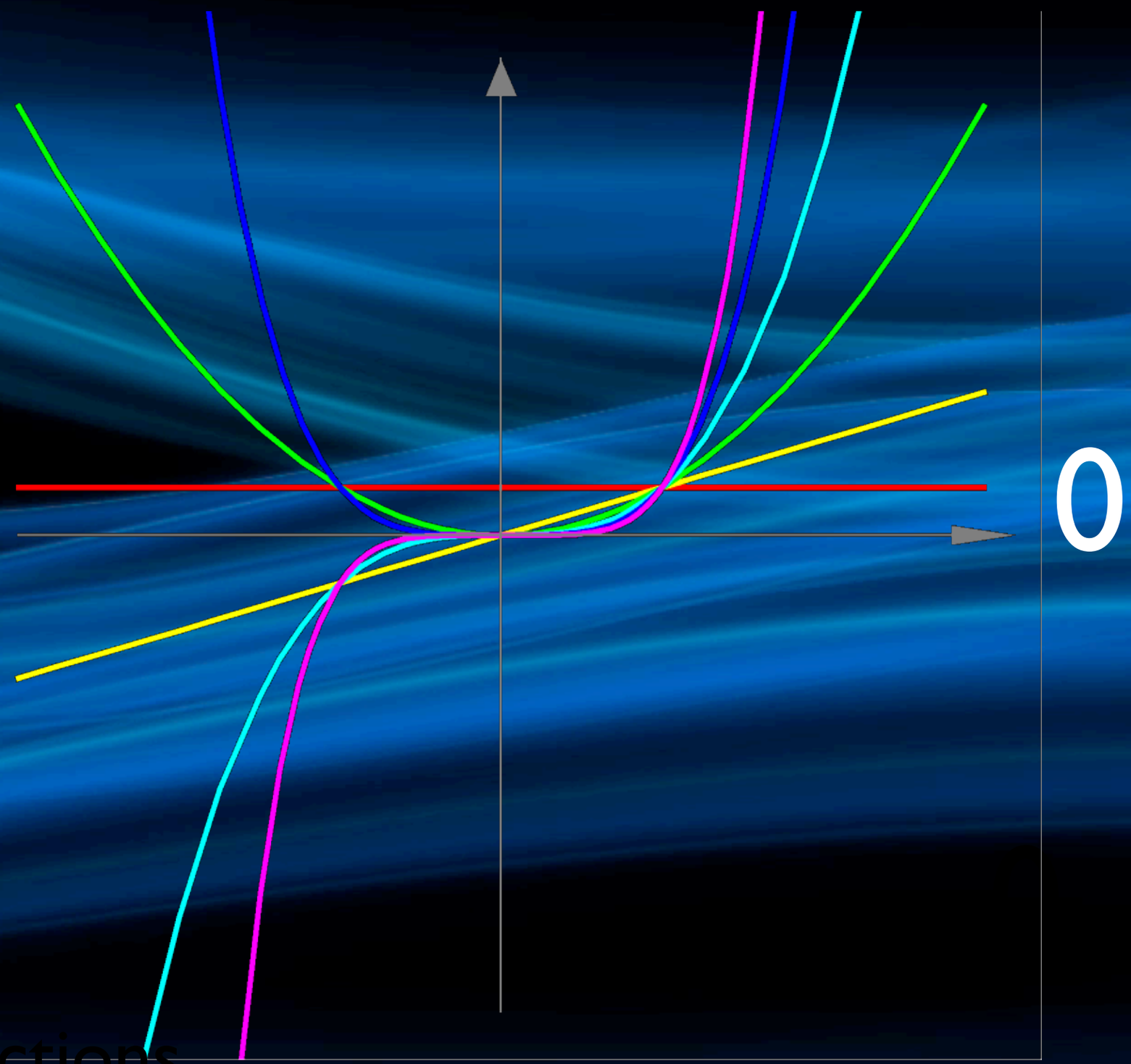
Linear Spaces



ker(A), im(A)


1
2
3
7
5
1
-1
-2
7
5
4
2
0
-2
-4
1
2
3
4
5
6
7
8
7
5
4
2
0
-2
-4
1
2
3
4
5
6
7
8

Random Variables Vectors



vectors functions

linear independence

independent random variables

The end

Dave Eggar, Floating, Album: Left of Blue