



# Linear Algebra

WITH APPLICATIONS

FOURTH EDITION

Otto Bretscher



---

# Linear Algebra with Applications

Fourth Edition

**Otto Bretscher**

Colby College

Prentice Hall  
is an imprint of

**PEARSON**

Upper Saddle River, New Jersey 07458

**Library of Congress Cataloging-in-Publication Data**

Bretscher, Otto.

Linear algebra with applications / Otto Bretscher.—4th ed.

p. cm.

Includes index.

ISBN 978-0-13-600926-9

I. Algebras, Linear—Textbooks. I. Title.

QA184.2.B73 2009

512'.5—dc22

2008028930

Editor-in-Chief, Collegiate Mathematics: *Deirdre Lynch*  
Senior Acquisitions Editor: *William Hoffman*  
Associate Editor/Print Supplements: *Caroline Celano*  
Project Manager: *Raegan Keida Heerema*  
Production Coordination/Composition: *ICC Macmillan Inc.*  
Associate Managing Editor: *Bayani Mendoza deLeon*  
Senior Managing Editor: *Linda Mihatov Behrens*  
Senior Operations Supervisor: *Alan Fischer*  
Operations Specialist: *Lisa McDowell*  
Executive Marketing Manager: *Jeff Weidenaar*  
Marketing Assistant: *Jon Connelly*  
Art Director: *Maureen Eide*  
Interior Designer: *Dina Curro*  
Cover Designer: *Daniel Conte*  
AV Project Manager: *Thomas Benfatti*  
Manager, Cover Visual Research & Permissions: *Karen Sanatar*  
Art Studio: *Laserwords Private Limited*  
Cover Image Credit: *Andy Burmfield/Istockphoto.com*

The cover is a close-up image of a guitar, showing four vibrating strings and the sound hole. Using Fourier analysis, we can measure the amplitudes of a plucked string's harmonics or even synthesize the guitar's sound. See pages 239–244.

**Prentice Hall**  
is an imprint of



© 2009, 2005, 2001, 1997 Pearson Education, Inc.  
Pearson Prentice Hall  
Pearson Education, Inc.  
Upper Saddle River, NJ 07458

All rights reserved. No part of this book may be reproduced, in any form or by any means, without permission in writing from the publisher.

Pearson Prentice Hall® is a trademark of Pearson Education, Inc.

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

ISBN-10: 0-13-600926-3

ISBN-13: 978-0-13-600926-9

Pearson Education Ltd., London  
Pearson Education Singapore, Pte. Ltd  
Pearson Education Canada, Inc.  
Pearson Education-Japan  
Pearson Education Australia PTY, Limited

Pearson Education North Asia, Ltd., Hong Kong  
Pearson Education de Mexico, S.A. de C.V.  
Pearson Education Malaysia, Pte. Ltd.  
Pearson Education Upper Saddle River, New Jersey

*To my parents  
Otto and Margrit Bretscher-Zwicky  
with love and gratitude*

# Contents

---

Text Features ix

Preface xi

<b>1</b>	<b>Linear Equations</b>	<b>1</b>
1.1	Introduction to Linear Systems	1
1.2	Matrices, Vectors, and Gauss–Jordan Elimination	8
1.3	On the Solutions of Linear Systems; Matrix Algebra	25
<b>2</b>	<b>Linear Transformations</b>	<b>40</b>
2.1	Introduction to Linear Transformations and Their Inverses	40
2.2	Linear Transformations in Geometry	54
2.3	Matrix Products	69
2.4	The Inverse of a Linear Transformation	79
<b>3</b>	<b>Subspaces of <math>\mathbb{R}^n</math> and Their Dimensions</b>	<b>101</b>
3.1	Image and Kernel of a Linear Transformation	101
3.2	Subspaces of $\mathbb{R}^n$ ; Bases and Linear Independence	113
3.3	The Dimension of a Subspace of $\mathbb{R}^n$	123
3.4	Coordinates	137
<b>4</b>	<b>Linear Spaces</b>	<b>153</b>
4.1	Introduction to Linear Spaces	153
4.2	Linear Transformations and Isomorphisms	165
4.3	The Matrix of a Linear Transformation	172
<b>5</b>	<b>Orthogonality and Least Squares</b>	<b>187</b>
5.1	Orthogonal Projections and Orthonormal Bases	187
5.2	Gram–Schmidt Process and $QR$ Factorization	203
5.3	Orthogonal Transformations and Orthogonal Matrices	210
5.4	Least Squares and Data Fitting	220
5.5	Inner Product Spaces	233

<b>6</b>	<b>Determinants</b>	<b>249</b>
6.1	Introduction to Determinants	249
6.2	Properties of the Determinant	261
6.3	Geometrical Interpretations of the Determinant; Cramer's Rule	277
<b>7</b>	<b>Eigenvalues and Eigenvectors</b>	<b>294</b>
7.1	Dynamical Systems and Eigenvectors: An Introductory Example	294
7.2	Finding the Eigenvalues of a Matrix	308
7.3	Finding the Eigenvectors of a Matrix	319
7.4	Diagonalization	332
7.5	Complex Eigenvalues	343
7.6	Stability	357
<b>8</b>	<b>Symmetric Matrices and Quadratic Forms</b>	<b>367</b>
8.1	Symmetric Matrices	367
8.2	Quadratic Forms	376
8.3	Singular Values	385
<b>9</b>	<b>Linear Differential Equations</b>	<b>397</b>
9.1	An Introduction to Continuous Dynamical Systems	397
9.2	The Complex Case: Euler's Formula	410
9.3	Linear Differential Operators and Linear Differential Equations	423
	<b>Appendix A Vectors</b>	<b>437</b>
	<b>Answers to Odd-Numbered Exercises</b>	<b>446</b>
	<b>Subject Index</b>	<b>471</b>
	<b>Name Index</b>	<b>478</b>

## SUBJECT INDEX

### A

- Affine system, 329
- Algebraic multiplicity of an eigenvalue, 312
  - and complex eigenvalues, 352
  - and geometric multiplicity, 326
- Algorithm, 18
- Alternating property of determinant, 251, 265
- Angle, 196
  - and orthogonal transformations, 211
- Approximations,
  - Fourier 232, 239
  - least-squares 17, 220
- Argument (of a complex number), 346
  - of a product, 348
- Associative law for matrix multiplication, 75
- Asymptotically stable equilibrium, 358
  - (see stable equilibrium)
- Augmented matrix, 11

### B

- Basis, 116, 158
  - and coordinates, 139
  - and dimension, 125, 159
  - and unique representation, 120
  - finding basis of a linear space, 161
  - of an image, 116, 128, 131
  - of a kernel, 127, 131
  - of  $\mathbb{R}^n$ , 132
  - standard, 126
- Binary digits, 112
- Block matrices, 75
  - determinant of, 257
  - inverse of, 87
  - multiplication of, 76
- Bounded trajectory, 363

### C

- Carbon dating, 407
- Cardano's formula, 316, 319, 356
- Cartesian coordinates, 10, 137
- Cauchy–Schwarz inequality, 195, 237
  - and angles, 196
  - and triangle inequality, 200
- Cayley–Hamilton Theorem, 332
- Center of mass, 21
- Change of basis, 177, 179
- Characteristic polynomial, 311
  - and algebraic multiplicity, 312

- and its derivative, 318
- of linear differential operator, 426
- of similar matrices, 326
- Chebyshev polynomials, 246
- Cholesky factorization, 383
- Circulant matrix, 356
- Classical adjoint, 287
- Closed-formula solution
  - for discrete dynamical system, 303
  - for inverse, 287
  - for least-squares approximation, 224
  - for linear system, 286
- Coefficient matrix
  - of a linear system, 11
  - of a linear transformation, 41
- Column of a matrix, 9
- Column space of a matrix, 105
- Column vector, 10
- Commuting matrices, 73
- Complements, 135
- Complex eigenvalues, 350
  - and determinant, 353
  - and rotation–scaling matrices, 352, 359, 416
  - and stable equilibrium, 359
  - and trace, 353
- Complex numbers, 155, 345
  - and rotation–scaling, 348
  - in polar form, 347
- Complex-valued functions, 411
  - derivative of, 411
  - exponential, 412
- Component of a vector, 10
- Composite functions, 69
- Computational complexity, 89, 96
- Concatenation, 323
- Conics, 23, 134
- Consistent linear system, 25
- Continuous least-squares condition, 238
- Continuous linear dynamical system, 400
  - stability of, 415
  - with complex eigenbasis, 416
  - with real eigenbasis, 405
- Contraction, 56
- Coordinates, 139, 158
- Coordinate transformation, 159, 167
- Coordinate vector, 139, 159
- Correlation (coefficient), 198
- Cramer's Rule, 286

- Cross product
  - and polar (QS) factorization, 394
  - in  $\mathbb{R}^3$ , 52, 443, 445
  - in  $\mathbb{R}^n$ , 275
  - rules for, 444
- Cubic equation (Cardano's formula), 316, 356
- Cubic splines, 19
- Cubics, 134
- Curve fitting
  - with conics, 23, 134
  - with cubics, 134
- D**
- Data compression, 393
- Data fitting, 225
  - multivariate, 228
- De Moivre's formula, 348
- Determinant, 85, 255
  - alternating property of, 251, 265
  - and characteristic polynomial, 310, 311
  - and eigenvalues, 315, 353
  - and Cramer's rule, 286
  - and elementary row operations, 265
  - and invertibility, 266
  - and Laplace expansion, 268, 270
  - and  $QR$  factorization, 279
  - as area, 85, 278
  - as expansion factor, 282
  - as volume, 280
  - is linear in rows and columns, 252, 262
  - of inverse, 268
  - of linear transformation, 272
  - of orthogonal matrix, 278
  - of permutation matrix, 260
  - of product, 267
  - of rotation matrix, 278
  - of similar matrices, 267
  - of  $3 \times 3$  matrix, 249
  - of transpose, 262
  - of triangular matrix, 257
  - of  $2 \times 2$  matrix, 85
  - patterns and, 255
  - Vandermonde, 273
  - Weierstrass definition of, 276
- Diagonalizable matrices, 333
  - and eigenbases, 333
  - and powers, 336
  - orthogonally, 367, 372
  - simultaneously, 342
- Diagonal matrix, 9
- Diagonal of a matrix, 9
- Dilation, 56
- Dimension, 125, 159
  - and isomorphism, 168
  - of image, 129, 165
  - of kernel, 129, 165
  - of orthogonal complement, 193
- Direction field, 401
- Discrete linear dynamical system, 303
  - and complex eigenvalues, 360, 362
  - and stable equilibrium, 358, 359
- Distance, 236
- Distributive Laws, 75
- Domain of a function, 43
- Dominant eigenvector, 405
- Dot product, 20, 28, 215, 441
  - and matrix product, 73, 215
  - and product  $A\vec{x}$ , 29
  - rules for, 442
- Dynamical system
  - (see continuous, discrete linear dynamical system)
- E**
- Eigenbasis, 322
  - and continuous dynamical system, 405
  - and diagonalization, 333
  - and discrete dynamical system, 303
  - and distinct eigenvalues, 324
  - and geometric multiplicity, 324
- Eigenfunction, 337, 426
- Eigenspaces, 320
  - and geometric multiplicity, 322
  - and principal axes, 381
- Eigenvalue(s), 300
  - algebraic multiplicity of, 312
  - and characteristic polynomial, 311
  - and determinant, 315, 353
  - and positive (semi)definite matrices, 379
  - and  $QR$  factorization, 316
  - and singular values, 387
  - and stable equilibrium, 359, 415
  - and trace, 315, 353
  - complex, 350, 362
  - geometric multiplicity of, 322
  - of linear transformation, 337
  - of orthogonal matrix, 302
  - of rotation-scaling matrix, 351
  - of similar matrices, 326
  - of symmetric matrix, 370
  - of triangular matrix, 310
  - power method for finding, 355

- Eigenvectors, 300  
 and linear independence, 323  
 dominant, 405  
 of symmetric matrix, 369
- Elementary matrix, 93
- Elementary row operations, 16  
 and determinant, 265  
 and elementary matrices, 93
- Ellipse, 69  
 as image of the unit circle, 69, 372, 387  
 as level curve of quadratic form, 381  
 as trajectory, 362, 416
- Equilibrium (state), 358
- Equivalence Relation, 146
- Error (in least-squares solution), 222
- Error-correcting codes, 112
- Euler identities, 240
- Euler's formula, 413
- Euler's theorem, 328
- Exotic operations (in a linear space), 172
- Expansion factor, 282
- Exponential functions, 398  
 complex-valued, 412
- F**
- Factorizations,  
 Cholesky, 383  
 $LDL^T$ , 220, 383  
 $LDU$ , 93, 220  
 $LL^T$ , 383  
 $LU$ , 93  
 $QR$ , 207, 208, 279, 316  
 $QS$ , 394  
 $SDS^{-1}$ , 336  
 $SDS^T$ , 367  
 $U\Sigma V^T$ , 390
- Field, 350
- Finite dimensional linear space, 162
- Flow line, 401
- Fourier analysis, 239
- Function, 43
- Fundamental theorem of algebra, 349
- Fundamental theorem of linear algebra, 129
- G**
- Gaussian integration, 246
- Gauss–Jordan elimination, 17  
 and determinant, 265  
 and inverse, 82
- Geometric multiplicity of an eigenvalue, 322  
 and algebraic multiplicity, 326  
 and eigenbases, 324
- Golden section, 331
- Gram–Schmidt process, 206  
 and determinant, 278  
 and orthogonal diagonalization, 372  
 and  $QR$  factorization, 207
- H**
- Harmonics, 241
- Hilbert space  $\ell_2$ , 201, 235  
 and quantum mechanics, 243
- Homogeneous linear system, 35
- Hyperbola, 381
- Hyperplane, 134
- I**
- Identity matrix  $I_n$ , 45
- Identity transformation, 45
- Image of a function, 101
- Image of a linear transformation,  
 105, 165  
 and rank, 129  
 is a subspace, 105, 165  
 orthogonal complement of, 221  
 written as a kernel, 111
- Image  
 of the unit circle, 69, 372, 387
- Imaginary part of a complex number, 346
- Implicit function theorem, 276
- Inconsistent linear system, 5, 17, 25  
 and least-squares, 222
- Indefinite matrix, 378
- Infinite dimensional linear space, 162
- Inner product (space), 234
- Input-output analysis, 5, 20, 90, 95, 150
- Intermediate value theorem, 68
- Intersection of subspaces, 123  
 dimension of, 135
- Invariant Subspace, 307
- Inverse (of a matrix), 80  
 and Cramer's rule, 287  
 determinant of, 268  
 of an orthogonal matrix, 215  
 of a product, 83  
 of a transpose, 216  
 of a  $2 \times 2$  matrix, 85  
 of block matrix, 87
- Inversion (in a pattern), 255
- Invertible function, 79
- Invertible matrix, 80  
 and determinant, 266  
 and kernel, 109
- Isomorphism, 167  
 and dimension, 168

**K**

- Kernel, 106, 165
  - and invertibility, 109
  - and linear independence, 120
  - and rank, 129
  - dimension of, 129
  - is a subspace, 108, 165
- Kyle numbers, 131, 320

**L**

- Laplace expansion, 268, 270
- $LDL^T$  factorization, 220, 383
- $LDU$  factorization, 93, 220
- Leading one, 16
- Leading variable, 13
- Least-squares solutions, 17, 201, 222
  - and normal equation, 223
  - minimal, 230
- Left inverse, 122
- Legendre polynomials, 246
- Length of a vector, 187, 442
  - and orthogonal transformations, 210
- Linear combination, 30, 156
  - and span, 105
- Linear dependence, 116
- Linear differential equation, 161, 423
  - homogeneous, 423
  - order of, 423
  - solution of, 425, 433
  - solving a first order, 431
  - solving a second order, 428, 429
- Linear differential operator, 423
  - characteristic polynomial of, 426
  - eigenfunctions of, 426
  - image of, 432
  - kernel of, 427
- Linear independence
  - and dimension, 126
  - and kernel, 120
  - and relations, 118
  - in  $\mathbb{R}^n$ , 116
  - in a linear space, 158
  - of eigenvectors, 323
  - of orthonormal vectors, 189
- Linearity of the determinant, 252, 262
- Linear relations, 118
- Linear space(s), 154
  - basis of, 158
  - dimension of, 159
  - finding basis of, 161
  - finite dimensional, 162
  - isomorphic, 167

**Linear system**

- closed-formula solution for, 286
  - consistent, 25
  - homogeneous, 35
  - inconsistent, 5, 25
  - least-squares solutions of, 222
  - matrix form of, 32
  - minimal solution of, 230
  - number of solutions of, 25
  - of differential equations: *see* continuous linear dynamical system
  - unique solution of, 27
  - vector form of, 32
  - with fewer equations than unknowns, 27
- Linear transformation, 41, 44, 48, 165
- image of, 105, 165
  - kernel of, 106, 165
  - matrix of, 41, 44, 47, 143, 173
- Lower triangular matrix, 9
- $LU$  factorization, 93
- and principal submatrices, 94

**M**

- Main diagonal of a matrix, 9
- Mass-spring system, 422
- Matrix, 9
  - (for a composite entry such as “zero matrix” *see* zero)
- Minimal solution of a linear system, 230
- Minors of a matrix, 269
- Modulus (of a complex number), 346
  - of a product, 348
- Momentum, 21
- Multiplication (of matrices), 72
  - and determinant, 267
  - column by column, 72
  - entry by entry, 73
  - is associative, 75
  - is noncommutative, 73
  - of block matrices, 76

**N**

- Negative feedback loop, 420
- Neutral element, 154
- Nilpotent matrix, 136, 375
- Norm
  - of a vector, 187
  - in an inner product space, 236
- Normal equation, 223
- Nullity, 129, 165
- Null space, 106

## O

- Orthogonal complement, 193
  - of an image, 221
- Orthogonally diagonalizable matrix, 367, 372
- Orthogonal matrix, 211, 216
  - determinant of, 278
  - eigenvalues of, 302
  - inverse of, 215
  - transpose of, 215
- Orthogonal projection, 58, 189, 191, 225, 237
  - and Gram–Schmidt process, 203
  - and reflection, 59
  - as closest vector, 222
  - matrix of, 217, 225
- Orthogonal transformation, 210
  - and orthonormal bases, 217
  - preserves dot product, 218
  - preserves right angles, 211
- Orthogonal vectors, 187, 236, 443
  - and Pythagorean theorem, 194
- Orthonormal bases, 189, 192
  - and Gram–Schmidt process, 203
  - and orthogonal transformations, 212
  - and symmetric matrices, 368
- Orthonormal vectors, 188
  - are linearly independent, 189
- Oscillator, 184

## P

- Parallelepiped, 280
- Parallel vectors, 440
- Parametrization (of a curve), 102
- Partitioned matrices, 76
  - (see block matrix)
- Pattern (in a matrix), 255
  - inversion in, 255
  - signature of, 255
- Permutation matrix, 89
  - determinant of, 260
- Perpendicular vectors, 187, 236, 443
- Phase portrait,
  - of continuous system, 400, 419
  - of discrete system, 304, 419
  - summary, 419
- Piecewise continuous function, 239
- Pivot, 16
- Polar factorization, 394
- Polar form (of a complex number), 347
  - and powers, 348
  - and products, 348
  - with Euler's formula, 413

- Positive (semi)definite matrix, 378
  - and eigenvalues, 379
  - and principal submatrices, 379
- Positively oriented basis, 290
- Power method for finding eigenvalues, 355
- Principal axes, 381
- Principal submatrices, 94
  - and positive definite matrices, 379
- Product  $A\vec{x}$ , 29
  - and dot product, 29
  - and matrix multiplication, 72
- Projection, 56, 67
  - (see also orthogonal projection)
- Pseudo-inverse, 230
- Pythagorean theorem, 194, 237

## Q

- $QR$  factorization, 207
  - and Cholesky factorization, 383
  - and determinant, 279
  - and eigenvalues, 316
  - is unique, 219
- $QS$  factorization, 394
- Quadratic forms, 377
  - indefinite, 378
  - negative (semi)definite, 378
  - positive (semi)definite, 378
  - principal axes for, 381
- Quaternions, 220, 356, 364
  - form a skew field, 356

## R

- Rank (of a matrix), 26, 129
  - and image, 129
  - and kernel, 129
  - and row space, 136
  - and singular values, 389
  - of similar matrices, 326
  - of transpose, 216
- Rank-nullity theorem, 129, 165
- Real part of a complex number, 346
- Reduced row-echelon form (rref), 16
  - and determinant, 265
  - and inverse, 80, 82
  - and rank, 26
- Redundant vector(s), 116, 158
  - image and, 116, 131
  - kernel and, 131
- Reflection, 59
- Regular transition matrix, 317, 354
- Relations (among vectors), 118
- Resonance, 184
- Riemann integral, 232, 234

- Rotation matrix, 62, 278
- Rotation-scaling matrix, 63
  - and complex eigenvalues, 351
  - and complex numbers, 348
- Row of a matrix, 9
- Row space, 136
- Row vector, 10
- Rule of 69, 408
  
- S**
- Sarrus's rule, 250
- Scalar, 12
- Scalar multiples of matrices, 28
- Scaling, 54
- Second derivative test, 383
- Secular equation, 308
- Separation of variables, 398, 407
- Shears, 64
- Signature of a pattern, 255
- Similar matrices, 145
  - and characteristic polynomial, 326
- Simultaneously diagonalizable matrices, 342
- Singular value decomposition (SVD), 390
- Singular values, 387
  - and ellipses, 387
  - and rank, 389
- Skew field, 356
- Skew-symmetric matrices, 214, 382
  - determinant of, 274
- Smooth function, 157
- Space of functions, 155
- Span
  - in  $\mathbb{R}^n$ , 105
  - in a linear space, 159
- Spectral theorem, 368
- Spirals, 344, 349, 362, 413, 416, 419
- Square matrix, 9
- Square-summable sequences, 201, 235
- Stable equilibrium, 358
  - of continuous system, 415
  - of discrete system, 359
- Standard basis, 126, 160
- State vector, 295
- Subspace
  - invariant, 307
  - of  $\mathbb{R}^n$ , 113
  - of a linear space, 157
- Sudoku, 75
- Sum of subspaces, 123
  - dimension of, 135
- Sums of matrices, 28
  
- Symmetric matrices, 214
  - are orthogonally diagonalizable, 368
  - have real eigenvalues, 370
  
- T**
- Target space (of a function), 43
- Tetrahedron, 92, 150, 289
- Theorem of Pythagoras, 194, 237
- Trace, 235, 310
  - and characteristic polynomial, 311
  - and eigenvalues, 315, 353
  - and inner product, 235
  - of similar matrices, 326
- Trajectory, 304, 400
  - bounded, 363
- Transpose (of a matrix), 214
  - and determinant, 262
  - and inverse, 216
  - and rank, 216
  - of an orthogonal matrix, 215
  - of a product, 216
- Triangle inequality, 200
- Triangular matrices, 9
  - and determinant, 257
  - and eigenvalues, 310
  - invertible, 89
- Triangularizable matrix, 375
- Trigonometric polynomials, 240
  
- U**
- Unit vector, 187, 443
- Upper triangular matrix, 9
  
- V**
- Vandermonde determinant, 273
- Vector(s), 10
  - angles between, 196
  - addition, 437
  - column, 10
  - coordinate, 139, 159
  - cross product of, 52, 249, 443
  - dot product of, 441
  - geometric representation of, 439
  - length of, 187, 442
  - orthogonal, 187, 443
  - parallel, 440
  - position, 439
  - redundant, 116
  - row, 10
  - rules of vector algebra, 438
  - scalar multiplication, 437
  - standard representation of, 10, 439
  - unit, 187, 443

velocity, 400  
 vs. points, 10, 439  
 zero, 438

Vector field, 401

Vector form of a linear system, 32

Vector space, 10, 154  
 (see linear space)

Velocity vector, 400

## W

Weierstrass definition of determinant, 276

Wigner semicircle distribution, 246

## X

Zero matrix, 9

Zero vector, 438

## NAME INDEX

- Abel, Niels Henrik, 316  
Alcuin of York, 24  
al-Kashi, Jamshid, 24  
al-Khowarizmi, Mohammed, 1, 18n  
al-Ma'mun, Caliph, 1  
Archimedes, 53, 331  
Argand, Jean Robert, 346n
- Banach, Stefan, 163  
Berlinski, David, 17  
Bézout, Etienne, 272  
Binet, Jacques, 272  
Burke, Kyle, 131
- Cajori, Florian, 272  
Cardano, Gerolamo, 316, 319  
Cauchy, Augustin-Louis, 195, 255n, 272, 309  
Cayley, Arthur, 272, 332  
Chebyshev, Pafnuty, 246  
Cholesky, André-Louis, 383  
Cramer, Gabriel, 135, 272, 286
- d'Alembert, Le Rond, 125, 309  
Dantzig, Tobias, 345  
De Moivre, Abraham, 348  
Descartes, René, 118n, 137, 155  
De Witt, Johan, 378
- Einstein, Albert, 356–357  
Euler, Leonhard Paul, 24, 135, 240, 243, 328, 345, 413
- Fermat, Pierre de, 137  
Fibonacci (Leonardo of Pisa), 330–331  
Fourier, Jean Baptiste Joseph, 239n
- Galois, Evariste, 316  
Gauss, Carl Friedrich, 17–18, 51, 246, 346n, 350  
Gram, Jörgen, 206  
Grassmann, Hermann Günther, 125
- Hamilton, William Rowan, 220, 332, 356  
Hamming, Richard W., 112  
Harriot, Thomas, 118n  
Heisenberg, Werner, 201, 244  
Helmholtz, Hermann von, 309  
Hilbert, David, 162, 201, 309
- Jacobi, Carl Gustav Jacob, 255n  
Jordan, Wilhelm, 17
- Kepler, Johannes, 233  
Kronecker, Leopold, 255n
- Lagrange, Joseph Louis, 309, 364  
Lanchester, Frederick William, 409n  
Laplace, Pierre-Simon Marquis de, 268n, 272  
Legendre, Adrien-Marie, 246  
Leibniz, Gottfried Wilhelm von, 272  
Leonardo da Vinci, 200  
Leontief, Wassily, 5, 150
- Mahavira, 24  
Mas-Colell, Andreu, 201  
Mazur, Barry, 345  
Medawar, Peter. B., 64  
Mumford, David, 423
- Newton, Isaac, 24
- Olbers, Wilhelm, 17
- Peano, Giuseppe, 154n  
Peirce, Benjamin, 413n  
Piazzi, Giuseppe, 17  
Pythagoras, 194, 243, 272
- Riemann, Bernhard, 232, 234, 246  
Roosevelt, Franklin D., 90n
- Samuelson, Paul E., 364  
Sarrus, Pierre Frédéric, 250n  
Schläfi, Ludwig, 126  
Schmidt, Erhard, 162, 206n  
Schrödinger, Erwin, 244  
Schwarz, Hermann Amandus, 195  
Seki, Kōwa (or Takakazu), 255, 272  
Simpson, Thomas, 246  
Strang, Gilbert, 393  
Strassen, Volker, 96  
Sylvester, James Joseph, 9n
- Thompson, d'Arcy, 64
- Vandermonde, Alexandre-Théophile, 255n, 274
- Weierstrass, Karl, 273, 276  
Wessel, Caspar, 346n  
Weyl, Hermann, 125  
Wigner, Eugene Paul, 246  
Wilson, Edward O., 285n