

Lecture 14: Basis

Recall that subset X of \mathbf{R}^n which is closed under addition and scalar multiplication is called a **linear subspace** of \mathbf{R}^n . We have to check three conditions: (a) $0 \in V$, (b) $\vec{v} + \vec{w} \in V$ if $\vec{v}, \vec{w} \in V$. (c) $\lambda \vec{v} \in V$ if \vec{v} and λ is a real number.

- 1 The image and kernel of a transformation are linear spaces. This is an important example since this is how we describe linear spaces, either as the image of a linear transformation or the kernel of a linear transformations. Both are useful and they are somehow dual to each other. The kernel is associated to row vectors because we are perpendicular to all row vectors, the image is associated to column vectors because we are perpendicular to all column vectors.

A set \mathcal{B} of vectors $\vec{v}_1, \dots, \vec{v}_m$ is called **basis** of a linear subspace X of \mathbf{R}^n if they are **linear independent** and if they **span** the space X . Linear independent means that there are no nontrivial **linear relations** $a_1\vec{v}_1 + \dots + a_m\vec{v}_m = 0$. Spanning the space means that very vector \vec{v} can be written as a linear combination $\vec{v} = a_1\vec{v}_1 + \dots + a_m\vec{v}_m$ of basis vectors.

- 2 Two nonzero vectors in the plane form a basis if they are not parallel.

- 3 The standard basis vectors $e_1 = \begin{bmatrix} 1 \\ 0 \\ \dots \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ \dots \\ 0 \end{bmatrix}$, \dots , $e_n = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 1 \end{bmatrix}$ form a basis in \mathbf{R}^n .

Given a basis \mathcal{B} in V . Every vector in V can be written in a **unique manner** as a linear combination of vectors in \mathcal{B} .

To see this, assume \vec{v} is written in two different ways

$$\vec{v} = a_1v_1 + a_2v_2 + \dots + a_nv_n = b_1v_1 + b_2v_2 + \dots + b_nv_n .$$

a Then $(a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \dots + (a_n - b_n)v_n = 0$. But this shows that the vectors are not linearly independent.

- 4 Given a probability space with 4 elements. Any random variable can be written in a unique way as a linear combination of the 4 random variables

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} .$$

In general, one a finite probability space. A basis defines n random variables such that every random variable X can be written as a linear combination of X_1, \dots, X_n .

A set of random variables X_1, \dots, X_n which form a basis on a finite probability space $\Omega = \{1, \dots, n\}$ describe everything. Every random variable X which we want to compute can be expressed using these random variables.

Given a bunch of vectors in a linear space V , we can construct a basis of V by sticking the vectors as columns into a matrix A , then pick the pivot columns of A .

- 5 Find a basis of the space V spanned by the three vectors

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} .$$

Solution: Form the matrix

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 5 \\ 3 & 9 & 1 \\ -2 & -6 & 0 \end{bmatrix} .$$

Row reduction shows that the first and third vector span the space V . This is a basis for V .

A $n \times n$ matrix $A = \begin{bmatrix} | & | & \dots & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \\ | & | & \dots & | \end{bmatrix}$ is invertible if and only if $\vec{v}_1, \dots, \vec{v}_n$ is a basis in \mathbf{R}^n .

- 6 Find a basis for the image and the kernel of $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. **Solution:** In reduced row

echelon form is $B = \text{rref}(A) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. To determine a basis of the kernel we write

$Bx = 0$ as a system of linear equations: $x + y = 0, z = 0$. The variable y is the free variable. With $y = t, x = -t$ is fixed. The linear system $\text{rref}(A)x = 0$ is solved by

$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. So, $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is a basis of the kernel. Because the first and third

vectors in $\text{rref}(A)$ are pivot columns, the vectors $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ form a basis of the

image of A .

Why do we not just always stick to the standard basis vectors $\vec{e}_1, \dots, \vec{e}_n$? The reason for the need of more general basis vectors is that they allow a **more flexible adaptation** to the situation. In geometry, the reflection of a ray at a plane or at a curve is better done in a basis adapted to the situation. For differential equations, the system can be solved in a suitable basis. Basis also matters in statistics. Given a set of random variables, we often can find a basis for them which consists of uncorrelated vectors.

How do we check that a set of vectors form a basis in \mathbf{R}^n ?

A set of n vectors $\vec{v}_1, \dots, \vec{v}_n$ in \mathbf{R}^n form a basis in \mathbf{R}^n if and only if the matrix A containing the vectors as column vectors is invertible.

7 The vectors $\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ form a basis of \mathbf{R}^3 because the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

is invertible.

More generally, the pivot columns of an arbitrary matrix A form a basis for the image of A . Since we represent linear spaces always as the kernel or image of a linear map, the problem of finding a basis to a linear space is always the problem of finding a basis for the image or finding a basis for the kernel of a matrix.

Homework due March 2, 2011

1 Find a basis for the image and kernel of the Chess matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

and verify the rank-nullity theorem in this case.

2 Find a basis for the set of vectors perpendicular to the image of A , where A is the Pascal matrix.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 0 & 3 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 6 & 0 & 4 & 0 & 1 \end{bmatrix}.$$

3 Verify that a vector is in the kernel of a matrix A^T if and only if it is perpendicular to the image of A .

Verify it in the following concrete example

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 6 & 0 \\ 3 & 9 & 1 \\ 4 & 1 & 1 \end{bmatrix}.$$

The matrix A^T denotes the matrix where the columns and rows of A are switched. It is called the transpose of the matrix A .