

Lecture 21: Midterm checklist

Probability theory

- Probability space** $(\Omega, \mathcal{A}, P) = (\text{laboratory}, \text{events}, \text{probability measure})$
- Random variable** A function from Ω to the reals.
- Data** $(x_1, x_2, \dots, x_n) = \text{Vector} \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} = \text{random variables over } \Omega = \{1, \dots, n\}$
- Event A or B** the intersection $A \cap B$.
- Event A and B** the union $A \cup B$.
- Not the event A** the complement $\Omega \setminus A$.
- Event A under the condition B.** $P[A|B] = P[A \cap B]/P[B]$
- Independent events** $P[A \cap B] = P[A] \cdot P[B]$.
- Independent random variables** $\{X \in [a, b]\}, \{Y \in [c, d]\}$ are independent events.
- Independence and correlation** Independent random variables are uncorrelated.
- Expectation** $E[X] = \sum_{\omega} X(\omega) = \sum_{x_i} x_i P[X = x_i]$
- Variance** $\text{Var}[X] = E[X^2] - E[X]^2$.
- Standard deviation** $\sigma[X] = \sqrt{\text{Var}[X]}$.
- Covariance** $\text{Cov}[X, Y] = E[XY] - E[X] \cdot E[Y]$.
- Correlation** $\text{Corr}[X, Y] = \text{Cov}[X, Y]/(\sigma[X]\sigma[Y])$.
- Uncorrelated** $\text{Corr}[X, Y] = 0$.
- Variance formula** $\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}[X, Y]$.
- Pythagoras** $\text{Var}[X] + \text{Var}[Y] = \text{Var}[X + Y]$ for uncorrelated random variables.
- Bayes formula** $P[A|B] = \frac{P[B|A] \cdot P[A]}{P[B|A] + P[B|A^c]}$.
- Bayes rule** $P[A_i|B] = \frac{P[B|A_i] \cdot P[A_i]}{\sum_{j=1}^n P[B|A_j] \cdot P[A_j]}$.
- Permutations** $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$ possibilities.
- Combinations** n^k possibilities to select from n and put back.
- Ordered selection** Choose k from n with order $n!/(n-k)!$
- Unordered selection** Choose k from n gives $\binom{n}{k} = n!/(k!(n-k)!)$
- Binomial distribution** $P[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$.
- Expectation of binomial distribution** pn .
- Variance of binomial distribution** $p(1-p)n$.
- For 0-1 data.** The expectation determines the variance.

Linear algebra

- Matrix A** is a $n \times m$ matrix, it has m columns and n rows, maps \mathbf{R}^m to \mathbf{R}^n .
- Square matrix** $n \times n$ matrix, maps \mathbf{R}^n to \mathbf{R}^n .
- Identity matrix** the diagonal matrix I_n satisfies $I_n v = v$ for all vectors v .
- Column Vector** $n \times 1$ matrix = column vector
- Row Vector** $1 \times n$ matrix = row vector.
- Linear transformation** $\vec{x} \mapsto A\vec{x}$, $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$, $T(\lambda\vec{x}) = \lambda T(\vec{x})$.
- Column vectors of A** are images of standard basis vectors $\vec{e}_1, \dots, \vec{e}_n$.
- Linear system of equations** $A\vec{x} = \vec{b}$, have n equations, m unknowns.
- Consistent system** $A\vec{x} = \vec{b}$: there is at least one solution \vec{x} .
- Vector form of linear equation** $x_1 \vec{v}_1 + \dots + x_n \vec{v}_n = \vec{b}$, \vec{v}_i columns of A .
- Matrix form of linear equation** $\vec{w}_i \cdot \vec{x} = b_i$, \vec{w}_i rows of A .
- Augmented matrix of** $A\vec{x} = \vec{b}$ is the matrix $[A|b]$ which has one column more as A .
- Coefficient matrix of** $A\vec{x} = \vec{b}$ is the matrix A .
- Matrix multiplication** $[AB]_{ij} = \sum_k A_{ik} B_{kj}$, dot i -th row with j 'th column.
- Gauss-Jordan elimination** $A \rightarrow \text{rref}(A)$ in row reduced echelon form.
- Gauss-Jordan elimination steps** SSS: Swapping, Scaling, Subtracting rows.
- Leading one** First nonzero entry in a row is equal to 1. Write $\boxed{1}$.
- Row reduced echelon form** (1) nonzero row has $\boxed{1}$, (2) columns with $\boxed{1}$ are zero except at $\boxed{1}$, (3) every row above row with $\boxed{1}$ has $\boxed{1}$ to the left.
- Pivot column** column with $\boxed{1}$ in $\text{rref}(A)$.
- Redundant column** column with no $\boxed{1}$ in $\text{rref}(A)$.
- Rank of matrix A** number of $\boxed{1}$ in $\text{rref}(A)$. It is equal to $\dim(\text{im}(A))$.
- Nulley of matrix A:** is defined as $\dim(\ker(A))$.
- Kernel of matrix** $\{\vec{x} \in \mathbf{R}^n, A\vec{x} = \vec{0}\}$.
- Image of matrix** $\{A\vec{x}, \vec{x} \in \mathbf{R}^n\}$.
- Inverse transformation of T** A transformation satisfying $S(T(x)) = x = T(S(x))$.
- Inverse matrix of A** Matrix $B = A^{-1}$ satisfies $AB = BA = I_n$
- Rotation in plane** $A = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$, rotate counter-clock by α .
- Dilation in plane** $\vec{x} \mapsto \lambda\vec{x}$, also called scaling. Given by diagonal $A = I_2$
- Rotation-Dilation** $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Scale by $\sqrt{a^2 + b^2}$, rotate by $\arctan(b/a)$.
- Reflection-Dilation** $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. Scale by $\sqrt{a^2 + b^2}$, reflect at line w, slope b/a .
- Horizontal and vertical shear** $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}$.
- Reflection about line** $\vec{x} \mapsto A\vec{x}$, $A = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$.
- Projection onto line** containing unit vector u : $A = \begin{bmatrix} u_1 u_1 & u_1 u_2 \\ u_2 u_1 & u_2 u_2 \end{bmatrix}$.

- Linear subspace** check $\vec{0} \in X$, $\vec{x}, \vec{y} \in X, \lambda \in \mathbf{R} \Rightarrow \vec{x} + \vec{y} \in X, \lambda \vec{x} \in X$.
- $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ **span** X : Every $\vec{x} \in X$ can be written as $\vec{x} = a_1 \vec{v}_1 + \dots + a_n \vec{v}_n$.
- $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ **linear independent** X : $\sum_i a_i \vec{v}_i = \vec{0}$ implies $a_1 = \dots = a_n = 0$.
- $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ **basis in** X : linear independent in X and span X .
- Dimension of linear space** X : number of basis elements of a basis in X .
- S-matrix** Coordinate transformation matrix containing basis vectors as columns.
- \mathcal{B} coordinates** $[\vec{v}]_{\mathcal{B}} = S^{-1} \vec{v}$, where $S = [\vec{v}_1, \dots, \vec{v}_n]$ contains basis vectors \vec{v}_i as columns.
- \mathcal{B} matrix** of T in basis \mathcal{B} . The matrix is $B = S^{-1} A S$.
- A **similar to** B : defined as $B = S^{-1} A S$. We write $A \sim B$.
- Row reduction** SSS: scale rows, swap rows and subtract row from other row.
- Row reduced echelon form** is a matrix in row reduced echelon form?
- Matrix-Transformation** The columns of A are the images of the basis vectors.
- Kernel-Image** Compute the kernel and the image by row reduction.
- System of linear equations** Solve a system of linear equation by row reduction.
- How many solutions** Are there 0, 1, ∞ solutions? $\text{rank}(A), \text{rank}[A, b]$ matter.
- Similar?** Check whether B^n, A^n are similar. Both invertible or not. Possibly find S .
- Linear space** 0 is in V , $v + w$ is in V and λv is in V .
- Linear transformation** is a given transformation linear or not?
- Space orthogonal to given space** write as row space of a matrix and find kernel.
- Number of solutions.** A linear system of equations has either exactly 0, 1 or ∞ many solutions.
- Solve system** Row reduce $[A|b]$ to get $[I_n|x]$ with solution x .
- Vectors perpendicular to a set of vectors,** get kernel of matrix which contains vectors as rows.
- Rank-nullity theorem** $\dim(\ker(A)) + \dim(\text{im}(A)) = m$, where A is $n \times m$ matrix.
- Number of basis elements** is independent of basis. Is equal to dimension.
- Basis of image of A** pivot columns of A form a basis of the image of A .
- Basis of kernel of A** introduce free variables for each redundant column of A .
- Inverse of 2×2 matrix** switch diagonal, change sign of wings and divide by det.
- Inverse of $n \times n$ matrix** Row reduce $[A|I_n]$ to get $[I_n|A^{-1}]$.
- Matrix algebra** $(AB)^{-1} = B^{-1}A^{-1}$, $A(B+C) = AB+AC$, etc. $AB \neq BA$ i.g.
- Invertible** $\Leftrightarrow \text{rref}(A) = I_n \Leftrightarrow$ columns form basis $\Leftrightarrow \text{rank}(A) = n, \Leftrightarrow \text{nullity}(A) = 0$.
- Similarity properties:** $A \sim B$ implies $A^n \sim B^n$. If A is invertible, B is invertible.
- Orthogonal vectors** $\vec{v} \cdot \vec{w} = 0$.
- length** $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$, **unit vector** \vec{v} with $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = 1$.
- Orthogonal basis** basis such that v_1, \dots, v_n are pairwise orthogonal, and length 1.
- Orthogonal complement of V** $V^\perp = \{v|v \text{ perpendicular to } V\}$.
- Projection onto V** orth. basis $P = QQ^T$ if Q has orthonormal columns.
- Orthogonal projection** onto V is $A(A^T A)^{-1} A^T$.
- Least square solution** of $A\vec{x} = \vec{b}$ is $\vec{x}_* = (A^T A)^{-1} A^T \vec{b}$.
- Data fitting** Find least square solution of equations when data are fitted exactly.