Lecture 25: More Determinants

In this lecture, we learn about a faster method to compute the determinant of a $n \times n$ matrix. Summing over all possible permutations is often not efficient. For a 20×20 matrix, we would already have to sum over $20! = 2432902008176640000 \sim 2.4 * 10^{18}$ entries. As a comparison, there are $4.3 * 10^{17}$ seconds (=13.7 billion years) since the big bang.

Linearity of the determinant

Lets take a general $n \times n$ matrix A. The following linearity property of determinants follows pretty much from the definition because for every pattern, the sum is right. The determinant is a sum over all patterns.

> The determinant is linear in every row and every column.

Lets see what this means for rows. For columns it is similar.

$$
\det\left(\begin{array}{ccccc} A_{11} & A_{12} & A_{13} & \ldots & A_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ v_1 & v_2 & v_3 & \ldots & v_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{n1} & A_{n2} & A_{n3} & \ldots & A_{nn} \end{array}\right) + \det\left(\begin{array}{ccccc} A_{11} & A_{12} & A_{13} & \ldots & A_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ w_1 & w_2 & w_3 & \ldots & w_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{n1} & A_{n2} & A_{n3} & \ldots & A_{nn} \end{array}\right)
$$

$$
= \det\left(\begin{array}{ccccc} A_{11} & A_{12} & A_{13} & \ldots & A_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ (v+w)_1 & (v+w)_2 & (v+w)_3 & \ldots & (v+w)_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{n1} & A_{n2} & A_{n3} & \ldots & A_{nn} \end{array}\right) .
$$

$$
\lambda \det\left(\begin{array}{ccccc} A_{11} & A_{12} & A_{13} & \ldots & A_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ v_1 & v_2 & v_3 & \ldots & v_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{n1} & A_{n2} & A_{n3} & \ldots & A_{nn} \end{array}\right) = \det\left(\begin{array}{ccccc} A_{11} & A_{12} & A_{13} & \ldots & A_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \lambda v_1 & \lambda v_2 & \lambda v_3 & \ldots & \lambda v_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{n1} & A_{n2} & A_{n3} & \ldots & A_{nn} \end{array}\right) .
$$

> Swapping two rows changes the sign of the determinant.

$$
\det\left(\begin{array}{ccccc} A_{11} & A_{12} & A_{13} & \ldots & A_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ v_1 & v_2 & v_3 & \ldots & v_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ w_1 & w_2 & w_3 & \ldots & w_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{n1} & A_{n2} & A_{n3} & \ldots & A_{nn} \end{array}\right) = -\det\left(\begin{array}{ccccc} A_{11} & A_{12} & A_{13} & \ldots & A_{1n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ w_1 & w_2 & w_3 & \ldots & w_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ v_1 & v_2 & v_3 & \ldots & v_n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A_{n1} & A_{n2} & A_{n3} & \ldots & A_{nn} \end{array}\right) .
$$

Proof. We have to show that the number of upcrossing changes by an odd number. Lets count the number of upcrossings before and after the switch. Assume row a and c are switched. We look at one pattern and assume that (a,b) be an entry on row a and (c,d) is an entry on row b. The entry (a,b) changes the number of upcrossings to (c,d) by 1 (there is one upcrossing from (a,b) to (c,d) before which is absent after).
For each entry (x,y) inside the rectangle (a,c) x (b,d), the number of upcrossings from and to (x,y) changes by two. (there are two upcrossings to and from the orange squares before which are absent after). For each entry outside the rectangle and different from (a,b),(c,d), the number of upcrossings does not change.
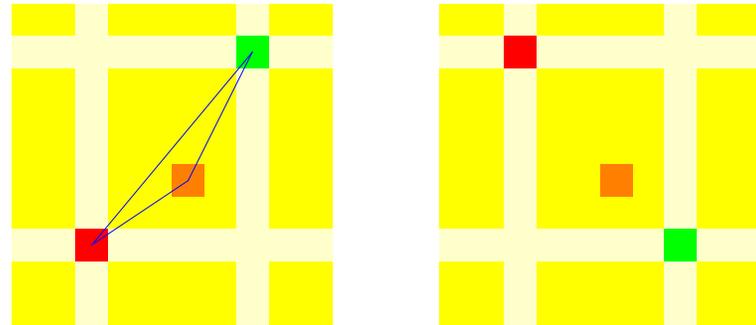
> It follows that if two rows are the same, then the determinant is zero.

Row reduction

We immediately get from the above properties what happens if we do row reduction. Subtracting a row from an other row does not change the determinant since by linearity we subtract the determinant of a matrix with two equal rows. Swapping two rows changes the sign and scaling a row scales the determinant.

> If $c_1, ..., c_k$ are the row reduction scale factors and m is the number of row swaps during row reduction, then
> $$ \det(A) = \frac{(-1)^m}{c_1 \cdots c_k} \det(rref(A)) . $$

Since row reduction is fast, we can compute the determinant of a 20×20 matrix in a jiffy. It takes about 400 operations and thats nothing for a computer.

1

$$
\begin{bmatrix} 4 & 1 & 1 & 1 \\ 4 & 2 & 2 & 2 \\ 4 & 1 & 6 & 3 \\ 4 & 1 & 1 & 7 \end{bmatrix} .
$$

Row reduce.

2 Compute the following determinant.

$$\det \begin{bmatrix} 0 & 2 & 5 & 6 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 3 & 2 \end{bmatrix} .$$

We could use the Laplace expansion or see that there is only one pattern. The simplest way however is to swap two rows to get an upper triangular matrix

$$\det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 6 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 24 .$$

3 The determinant of

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

is -1 because swapping the last row to the first row gives the identity matrix. Alternatively we could see that this permutation matrix has 5 upcrossings and that the determinant is -1.

A matrix is invertible if and only if $\det(A) \neq 0$.

Product of matrices

One of the main reasons why determinants are interesting is because of the following property

$\det(A \cdot B) = \det(A)\det(B)$

Proof. One can bring the $n \times n$ matrix $[A|AB]$ into row reduced echelon form. Similar than the augmented matrix $[A|b]$ was brought into the form $[1|A^{-1}b]$, we end up with $[1|A^{-1}AB] = [1|B]$. By looking at the $n \times n$ matrix to the left during the Gauss-Jordan elimination process, the determinant has changed by a factor $\det(A)$. We end up with a matrix B which has determinant $\det(B)$. Therefore, $\det(AB) = \det(A)\det(B)$.

$\det(A^T) = \det(A)$

Proof. Every upcrossing is a pair of entries A_{ij}, A_{kl} where $k > i, l > j$. If we look at the transpose, this pair of entries appears again as an upcrossing. So, every summand in the permutation definition of the determinant appears with the same sign also in the determinant of the transpose.

What are determinants useful for?

As the name tells, determinants determine a lot about matrices. We can see from the determinant whether the matrix is invertible.
An other reason is that determinants allow explicit formulas for the inverse of a matrix. We might look at this next time. Next week we will see that determinants allow to define the characteristic polynomial of a matrix whose roots are the important eigenvalues. In analysis, the determinant appears in change of variable formulas:

$$\int_S f(x) \, dx = \int_{u(S)} f(y) \, |\det(Du^{-1}(y)|dy .$$

Physicists are excited about determinants because summation over all possible "paths" is used as a quantization method. The Feynmann path integral is a "summation" over a suitable class of paths and leads to quantum mechanics. The relation with determinants comes because each summand in a determinant can be interpreted as a contribution of a path in a finite graph with n nodes.

Homework due April 6, 2011

1 Find the determinant of

$$\begin{bmatrix} 3 & 2 & 0 & 0 & 0 & 0 \\ 3 & 3 & 2 & 0 & 0 & 0 \\ 3 & 3 & 3 & 2 & 0 & 0 \\ 3 & 3 & 3 & 3 & 2 & 0 \\ 3 & 3 & 3 & 3 & 3 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 \end{bmatrix}$$

2 Find the determinant of

$$\begin{bmatrix} 3 & 1 & 1 & 2 & 2 & 2 \\ 1 & 0 & 1 & 2 & 2 & 2 \\ 1 & 0 & 3 & 2 & 2 & 2 \\ 0 & 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 & 1 & 4 \end{bmatrix}$$

3 a) Find an example showing that $\det(A + B) \neq \det(A) + \det(B)$.
b) How do you modify $\det(\lambda A) = \lambda\det(A)$ to make it correct if A is a $n \times n$ matrix?