

## Lecture 29: Eigenvectors

### Eigenvectors

Assume we know an eigenvalue  $\lambda$ . How do we compute the corresponding eigenvector?

The **eigenspace** of an eigenvalue  $\lambda$  is defined to be the linear space of all eigenvectors of  $A$  to the eigenvalue  $\lambda$ .

The eigenspace is the kernel of  $A - \lambda I_n$ .

Since we have computed the kernel a lot already, we know how to do that.

The dimension of the eigenspace of  $\lambda$  is called the **geometric multiplicity** of  $\lambda$ .

Remember that the multiplicity with which an eigenvalue appears is called the algebraic multiplicity of  $\lambda$ :

The algebraic multiplicity is larger or equal than the geometric multiplicity.

Proof. Let  $\lambda$  be the eigenvalue. Assume it has geometric multiplicity  $m$ . If  $v_1, \dots, v_m$  is a basis of the eigenspace  $E_\mu$  form the matrix  $S$  which contains these vectors in the first  $m$  columns. Fill the other columns arbitrarily. Now  $B = S^{-1}AS$  has the property that the first  $m$  columns are  $\mu e_1, \dots, \mu e_m$ , where  $e_i$  are the standard vectors. Because  $A$  and  $B$  are similar, they have the same eigenvalues. Since  $B$  has  $m$  eigenvalues  $\lambda$  also  $A$  has this property and the algebraic multiplicity is  $\geq m$ .

You can remember this with an analogy: the **geometric mean**  $\sqrt{ab}$  of two numbers is smaller or equal to the **algebraic mean**  $(a+b)/2$ .

- 1 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

This matrix has a large kernel. Row reduction indeed shows that the kernel is 4 dimensional. Because the algebraic multiplicity is larger or equal than the geometric multiplicity there are 4 eigenvalues 0. We can also immediately get the last eigenvalue from the trace 15. The eigenvalues of  $A$  are 0, 0, 0, 0, 15.

- 2 Find the eigenvalues of  $B$ .

$$B = \begin{bmatrix} 101 & 2 & 3 & 4 & 5 \\ 1 & 102 & 3 & 4 & 5 \\ 1 & 2 & 103 & 4 & 5 \\ 1 & 2 & 3 & 104 & 5 \\ 1 & 2 & 3 & 4 & 105 \end{bmatrix}$$

This matrix is  $A + 100I_5$  where  $A$  is the matrix from the previous example. Note that if  $Bv = \lambda v$  then  $(A + 100I_5)v = \lambda + 100)v$  so that  $A, B$  have the same eigenvectors and the eigenvalues of  $B$  are 100, 100, 100, 100, 115.

- 3 Find the determinant of the previous matrix  $B$ . **Solution:** Since the determinant is the product of the eigenvalues, the determinant is  $100^4 \cdot 115$ .

- 4 The shear  $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$  has the eigenvalue 1 with algebraic multiplicity 2. The kernel of  $A - I_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  is spanned by  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and the geometric multiplicity is 1.

- 5 The matrix  $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$  has eigenvalue 1 with algebraic multiplicity 2 and the eigenvalue 0 with multiplicity 1. Eigenvectors to the eigenvalue  $\lambda = 1$  are in the kernel of  $A - 1$  which is the kernel of  $\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$  and spanned by  $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ . The geometric multiplicity is 1.

If all eigenvalues are different, then all eigenvectors are linearly independent and all geometric and algebraic multiplicities are 1. The eigenvectors form then an eigenbasis.

Proof. If all are different, there is one of them  $\lambda_i$  which is different from 0. We use induction with respect to  $n$  and assume the result is true for  $n-1$ . Assume that in contrary the eigenvectors are linearly dependent. We have  $v_i = \sum_{j \neq i} a_j v_j$  and  $\lambda_i v_i = Av_i = A(\sum_{j \neq i} a_j v_j) = \sum_{j \neq i} a_j \lambda_j v_j$  so that  $v_i = \sum_{j \neq i} b_j v_j$  with  $b_j = a_j \lambda_j / \lambda_i$ . If the eigenvalues are different, then  $a_j \neq b_j$  and by subtracting  $v_i = \sum_{j \neq i} a_j v_j$  from  $v_i = \sum_{j \neq i} b_j v_j$ , we get  $0 = \sum_{j \neq i} (b_j - a_j) v_j = 0$ . Now  $(n-1)$  eigenvectors of the  $n$  eigenvectors are linearly dependent. Now use the induction assumption.

Here is an other example of an eigenvector computation:

- 6 Find all the eigenvalues and eigenvectors of the matrix

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

**Solution.** The characteristic polynomial is  $\lambda^4 - 1$ . It has the roots  $1, -1, i, -i$ . Instead of computing the eigenvectors for each eigenvalue, write

$$v = \begin{bmatrix} 1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix},$$

and look at  $Bv = \lambda v$ .

Where are eigenvectors used: in class we will look at some applications: **Hückel theory**, **orbitals** of the Hydrogen atom and **Page rank**. In all these cases, the eigenvectors have immediate interpretations. We will talk about page rank more when we deal with Markov processes.

The page rank vector is an eigenvector to the Google matrix.

These matrices can be huge. The google matrix is a  $n \times n$  matrix where  $n$  is larger than 10 billion!

<sup>1</sup>The book of Lanville and Meyher of 2006 gives 8 billion. This was 5 years ago.

## Homework due April 13, 2011

1 Find the eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -2 \\ 3 & 6 & -5 \end{bmatrix}.$$

2 a) Find the eigenvectors of  $A^{10}$ , where  $A$  is the previous matrix.  
b) Find the eigenvectors of  $A^T$ , where  $A$  is the previous matrix.

3 This is homework problem 40 in section 7.3 of the Bretscher book.



Photos of the Swiss lakes in the text. The pollution story is fiction fortunately.



The vector  $A^n(x)b$  gives pollution levels in the Silvaplana, Sils and St Moritz lake  $n$  weeks after an oil spill. The matrix is  $A = \begin{bmatrix} 0.7 & 0 & 0 \\ 0.1 & 0.6 & 0 \\ 0 & 0.2 & 0.8 \end{bmatrix}$  and  $b = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$  is the initial pollution level. Find a closed form solution for the pollution after  $n$  days.