3/23-25/2011: Midterm exam

Your Name:

- Please enter your name in the above box. You have to do this exam on your own.

- Answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.

- Do not detach pages from this exam packet or unstaple the packet.

- Please write neatly Answers which are illegible for the grader can not be given credit.

- Unless stated, give computations, details reasons.

- No calculators, computers, or other electronic aids are allowed. There is no hour time limit. 90 minutes should suffice and 3 hours are reasonable. You have to chose one day from the 23-25th and do the exam during that day. Staple it together and bring it to class on Friday.

- The exam is open book in the sense that you can use the class lecture notes and any handwritten notes of yours but no other books or text resources.

1		10
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		100

Problem 1) TF questions (10 points)

Solution:
All the TF questions are true except, 1,7,9

1) ☐ T ☐ F   Two events A, B which are disjoint satisfy $P[A \cap B] = P[A] + P[B]$.
Explanation:

Solution:
It is the union, not the intersection.

2) ☐ T ☐ F   Two events A, B which are independent satisfy $P[A \cap B] = P[A] \cdot P[B]$.
Explanation:

Solution:
By definition

3) ☐ T ☐ F   The empty set is always an event in any probability space.
Explanation:

Solution:
By definition, the entire probability space is there and so also the complement.

4) ☐ T ☐ F   If T is a linear map which satisfies $T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $T(\begin{bmatrix} 1 \\ -1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, then T is orthogonal.
Explanation:

Solution:
Use the vectors as a basis. They form an orthogonal system.

5) ☐ T ☐ F   The vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are linearly independent.

Explanation:

Solution:
One can see this from the formula for example. Since $(A^T A)^{-1} = A^{-1} A^T$, one has $(A^T A)^{-1} A^T = A^{-1}$.

Solution:
row reduce

6) T F   The expectation of the random variable $X = (2, 3, 5)$ is $10/3$.

Explanation:

Solution:
It is 2+3+5 divided by 3.

7) T F   The standard deviation of the random variable $X = (5, 6, 7)$ is $1/3$.

Explanation:

Solution:
The mean is 6 so that $X - E[X] = (-1, 0, 1)$ which has length $\sqrt{2}$.

8) T F   Let A, B be arbitrary 2×2 matrices. If a vector x is in the kernel of A, then it is in the kernel of BA.

Explanation:

Solution:
$Ax = 0$ implies $BAx = 0$. But ABx is not necessarily 0.

9) T F   Let A, B be arbitrary 2×2 matrices. If a vector x is in the kernel of B, then it is in the kernel of BA.

Explanation:

Solution:
$Ax = 0$ implies $BAx = 0$. But ABx is not necessarily 0.

10) T F   The least square solution of the linear system of equations $Ax = y$ is a real unique solution of the system if A is invertible.

Problem 2) (10 points)

Match the matrices with their actions:

A-J	domain	codomain	A-J	domain	codomain

$A=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$B=\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$

$C=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$D=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$E=\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

$F=\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$

$G=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$H=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$I=\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$J=\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

Solution:
$B, F, A, G, C, H, D, I, J, E$

Problem 3) (10 points) Systems of linear equations

a) (6 points) Find the general solution of the following system of linear equations using row reduction.

$$\begin{aligned} x+y+z+u+v &= 4 \\ x-y+z-u+v &= 0 \\ x-y+z &= 2 \end{aligned}$$

b) (2 points) The solutions $\vec{x} = \begin{bmatrix} x \\ y \\ z \\ u \\ v \end{bmatrix}$ can be written as $\vec{x} = \vec{b} + V$, where $\vec{b}$ is a particular solution and V is a linear space. What is the dimension of V?

c) (2 points) Which of the three cases did appear: exactly one solution, no solution or infinitely many solutions?

Solution:
Row reduce the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 2 \end{bmatrix}.$$

It becomes

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 2 \end{bmatrix}.$$

a) Write the system with the row reduced matrix and introduce free variables. The solution can be written as $[2, 0, 0, 2, 0]^T + s[-1, 0, 1, 0, 0]^T + t[-1, 1, 0, 1, 1]^T$, where s and t are free variables.

b) The dimension of the solution space is $\boxed{2}$. This is the dimension of the kernel of the matrix A which appears in the equation $Ax = b$.

c) The system is consistent. But we have $\boxed{\text{infinitely many solutions}}$.

Problem 4) (10 points) Random variables, independence

Lets look at the following two vectors in R^4:

$$X = \begin{bmatrix} 7 \\ -5 \\ -4 \\ 2 \end{bmatrix}, Y = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}.$$

a) (2 points) The two vectors span a linear space V. Write V it as an image of a matrix A.

b) (3 points) Find the space W of all vectors perpendicular to X, Y. Write W as the kernel of a matrix B.

We can see these matrices also as random variables over the probability space $\{1, 2, 3, 4\}$. For example $X(1) = 7, X(2) = -5, X(3) = -4, X(4) = 2$.

c) (2 points) Check that the two random variables X, Y are uncorrelated. What does Pythagoras tell about $\text{Var}[X + Y]$?

d) (3 points) Are the two random variables X, Y independent random variables?

Solution:

a) $A = \begin{bmatrix} 7 & 1 \\ -5 & -1 \\ -4 & 2 \\ 2 & -2 \end{bmatrix}$.

b) $B = \begin{bmatrix} 7 & -5 & -4 & 2 \\ 1 & -1 & 2 & -2 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & -9 & 8 \end{bmatrix}$. Its kernel is spanned by $[7, 9, 1, 0]^T$ and $[-6, -8, 0, 1]^T$. c) $Var[X + Y] = \text{Var}[X] + \text{Var}[Y]$.
d) No, $P[X = 7, Y = 1] = P[X = 7]P[Y = 1]$ is not true.

Problem 5) (10 points) Basis and Image

a) (5 points) Find a basis for the kernel of the following **diamond matrix**:

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 8 & 8 & 0 & 0 & 0 \\ 0 & 0 & 8 & 8 & 8 & 8 & 8 & 0 & 0 \\ 0 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 0 \\ 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\ 0 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 0 \\ 0 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 0 \\ 0 & 0 & 0 & 8 & 8 & 8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

b) (5 points) Find a basis for the image of the matrix A.

Solution:
Row reduce the matrix A to get

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We see that the first 5 columns are pivot columns and that the last 4 columns are redundant columns.
a) The 4 vectors

$$\mathcal{B}_{Ker} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$$

form a basis for the kernel.
b) The first 5 column vectors of the matrix A form a basis for the image:

$$\mathcal{B}_{Ran} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 8 \\ 8 \\ 8 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 8 \\ 8 \\ 8 \\ 8 \\ 8 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ 8 \\ 8 \\ 8 \\ 8 \\ 8 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ 8 \\ 8 \\ 8 \\ 8 \\ 8 \\ 8 \\ 8 \\ 8 \end{bmatrix} \right\}$$

Problem 6) (10 points) Inverse and Coordinates

Let

$$S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

a) (4 points) Find the inverse of S by row reducing the 2×4 matrix $[S|I_2]$.

b) (2 points) Assume the matrix A describes the transformation in the standard basis. Find the matrix B in the basis given by the column vectors of S.

c) (2 points) Two of the three matrices A, B, S are similar. Which ones?

Solution:
a) $S^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} /2.$
b) $B = S^{-1}AS = S$ because $A = S$.
c) All three matrices A, B, S are similar.

Problem 7) (10 points) Expectation, Variance, Covariance

The vectors $X = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 5 \\ 7 \end{bmatrix}$ and $Y = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$ can be seen as random variables over a probability space with 6 elements. They encode two different data measurements, where 6 probes were taken.

a) (2 points) Find the expectations $E[X], E[Y]$.

b) (2 points) Find the variances $\text{Var}[X]$ and $\text{Var}[Y]$.

c) (2 points) Find the standard deviations $\sigma[X]$ and $\sigma[Y]$.

d) (2 points) What is the covariance $\text{Cov}[X, Y]$ of X and Y?

e) (2 points) Find the correlation $\text{Corr}[X, Y]$ of X and Y.

Solution:
a) $E[X] = 19/6, E[Y] = 1/2.$
b) $\text{Var}[X] = E[X^2] - E[X^2] = 173/36$
$\text{Var}[Y] = E[Y^2] = E[Y^2] = 5/4$
c) $\sigma[X] = \sqrt{173}/6, \sigma[Y] = \sqrt{5}/2.$
d) $\text{Cov}[X, Y] = (E[XY] - E[X]E[Y]) = -23/12.$
e) $\text{Corr}[X, Y] = (E[XY] - E[X]E[Y])/(\sigma[X]\sigma[Y]) = (-23/12)/Sqrt[173 * 5/(36 * 4)] = -23/\sqrt{865}.$

Problem 8) (10 points) Combinatorics and Binomial Distribution

a) (2 points) We throw 20 coins. What is the probability that 5 heads show up?

b) (3 points) We throw 7 dice. What is the probability that 3 of the dice show the number 5?

c) (5 points) Assume that the random variable X counts the number of heads when throwing a coin 20 times. What is the expectation of this random variable?

Solution:
a) $\begin{pmatrix} 20 \\ 5 \end{pmatrix} (1/2)^5 (1/2)^5.$

b) $\begin{pmatrix} 7 \\ 3 \end{pmatrix} (1/6)^3 (5/6)^4.$

c) $20 * 1/2 = 10.$

Problem 9) (10 points) Bayes formula

a) (7 points) We throw 10 coins. What is the probability that the first coin is head if we know that 5 times heads comes up?

b) (3 points) Is the following argument correct? Whether your answer is "yes" or "no", give a short reason. The chance that an earth quake hit is $P[A] = 1/1000$. The chance that a tsunami hits is $P[B] = 1/1000$. Therefore, the chance that an earth quake and a tsunami hit both is $P[A] \times P[B] = 1/1'000'000$. The event that a quake and tsunami hit at the same time is therefore a one in a million event.

Solution:
a) $A = \{X_1 = 1\}$ $B = \{X = 5\}$. We use the **Bayes formula**

$$P[A|B] = P[B|A]P[A]/(P[B|A] + P[B|A^c]) .$$

$P[B|A] = B(9, 4)(1/2)^4(1/2)^5$. $P[B|A^c] = B(10, 5)(1/2)^5(1/2)^4$. $P[A] = 1/2$. The result is 1/2.
b) The two events are not necessarily independent. Actually they are often correlated since Earthquakes and Tsunamis can occur together if the epicenter is in the ocean.

Problem 10) (10 points) Data fitting

Fit the following data using functions of the form $f(x) = ax + bx^3$.

x	y
1	0
1	1
1	2
-1	1

Solution:

Write down the system of equations

$$
\begin{aligned}
a + b &= 0 \\
a + b &= 1 \\
a + b &= 2 \\
-a - b &= 1
\end{aligned}
$$

This is in matrix form $Ax = b$, where

$$
A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}.
$$

Now form $A^T A = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$ and $A^T b = \{2, 2\}$. The matrix $A^T A$ is not invertible. Indeed, we could have seen that the kernel of A is nontrivial. It is a rare case, where the fitting problem does not have a unique solution. The figure shows what is going on. There are many cubic curves of this type which have the same least square property. It is the three data points which are on top of each other which produced this ambiguity. Moving just one data point a bit away, would have produced us a unique solution.