1. PDF
2. Normal
3. CDF
4. Mean

Unit 23

PDF and CDF

1 PDF

$f(x)$ piecewise continuous

- $f(x) \geq 0$
- $\int_{-\infty}^{\infty} f(x)\,dx = 1$

a)

$e^{-x} = f(x)$
$x \geq 0$

$f(x) = 0$
else

exponential distribution

Check: $f(x) \geq 0$

$\int^{\infty} f(x)\,dx$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} e^{-x}\,dx$$

$$-e^{-x}\Big|_0^{\infty} = -e^{-\infty} + e^{-0} = \boxed{1}$$

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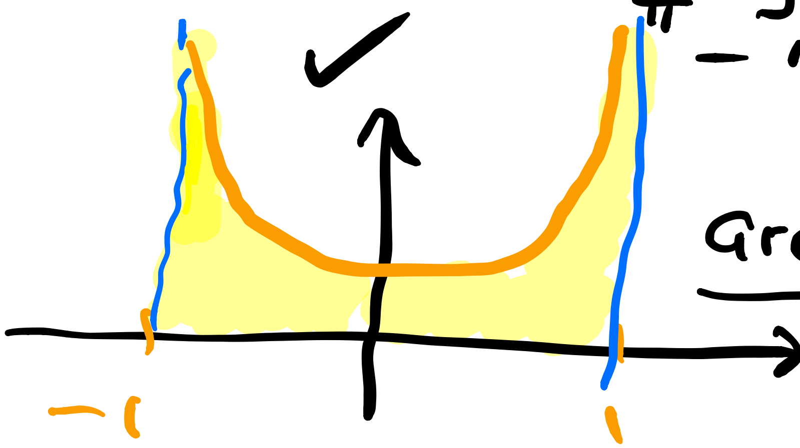
b) $f(x) = \dfrac{1}{\pi\sqrt{1-x^2}}$ on $[-1,1]$

$f(x) = 0$ else

In homework: $\dfrac{1}{\pi}\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx = 1$ ✓



arcsin distribution

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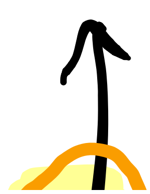
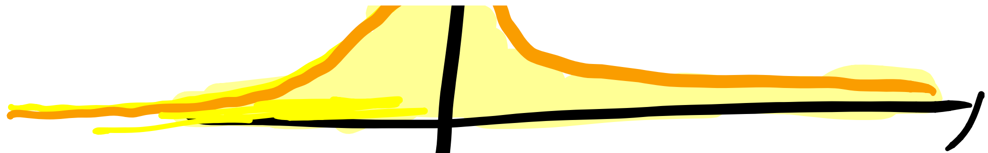
c) $f(x) = \dfrac{1}{1+x^2}\,\dfrac{1}{\pi}$ Cauchy distribution

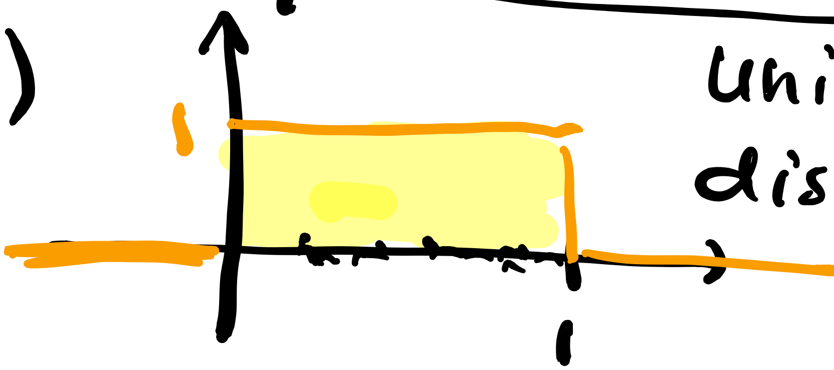
is a PDF





High Risk events.

a)



uniform distribution

Mathematica: Random[]

② Normal distribution

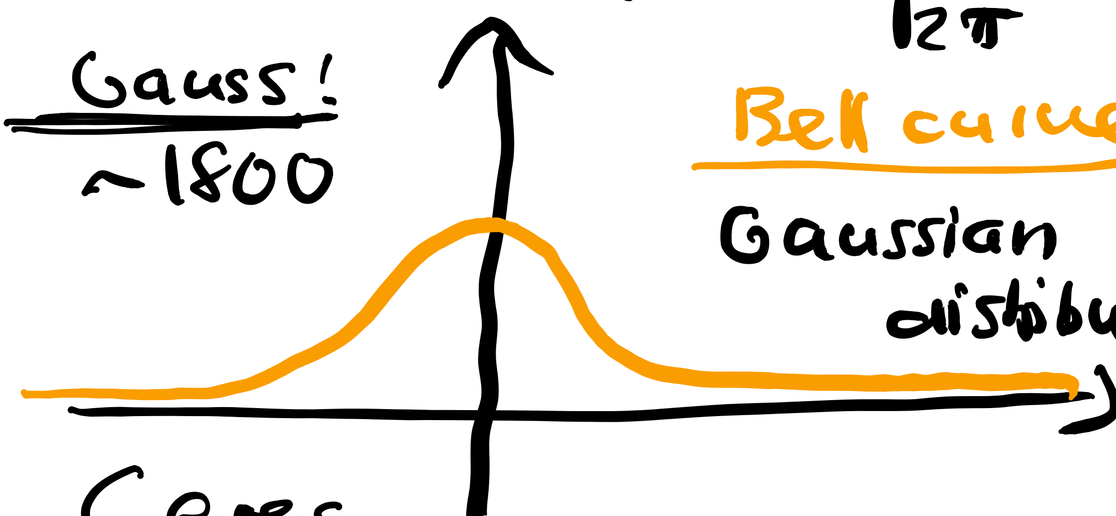
PDF: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

Gauss!

~1800

Bell curve

Gaussian distribution



Ceres

Challenge: verify

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}}$$

③ $\boxed{\text{CDF}}$

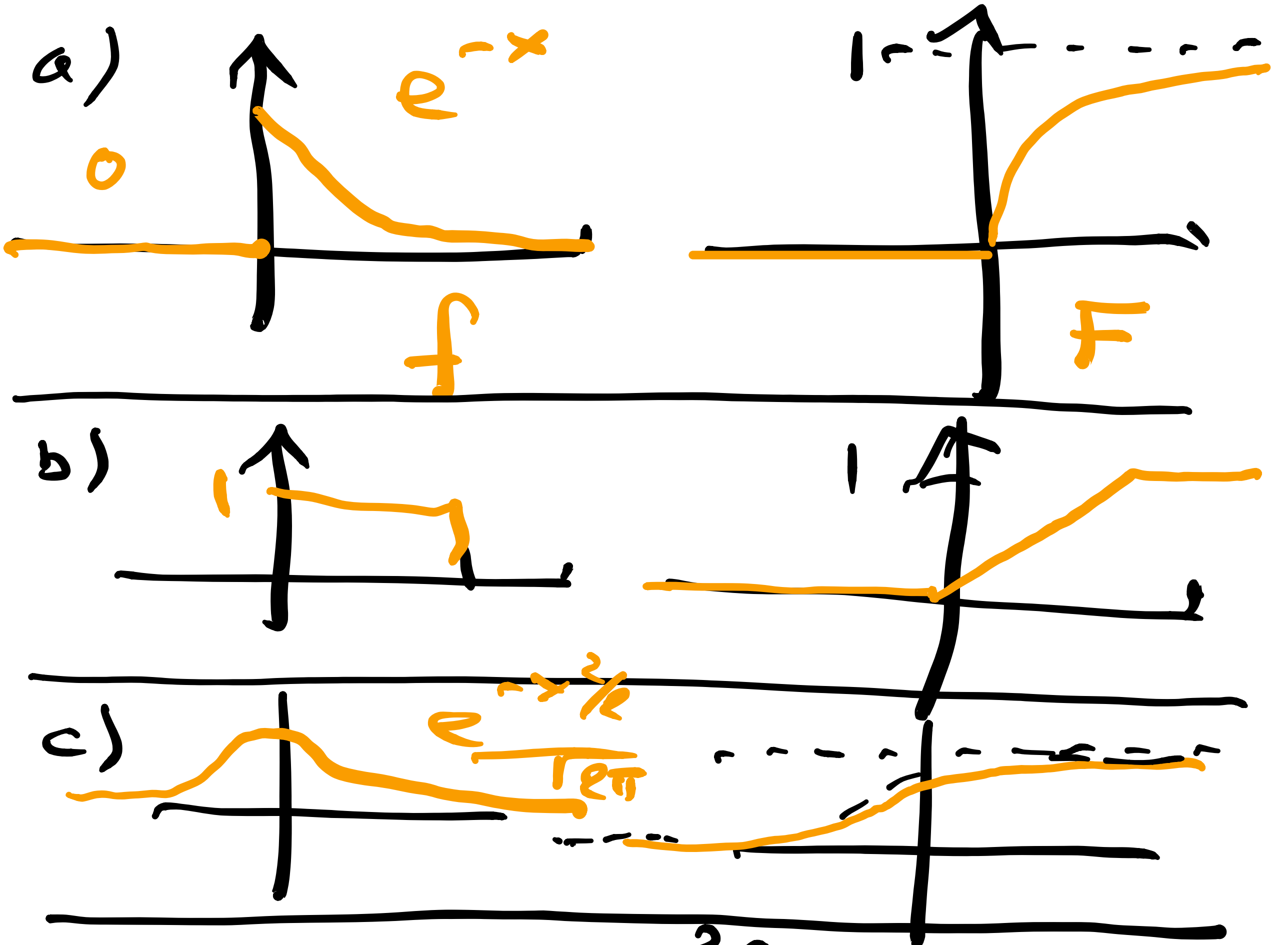
Cumulative
Distribution
Function

f is a PDF.

$$\boxed{\int_{-\infty}^{x} f(t)\,dt = F(x)}$$

antiderivative.

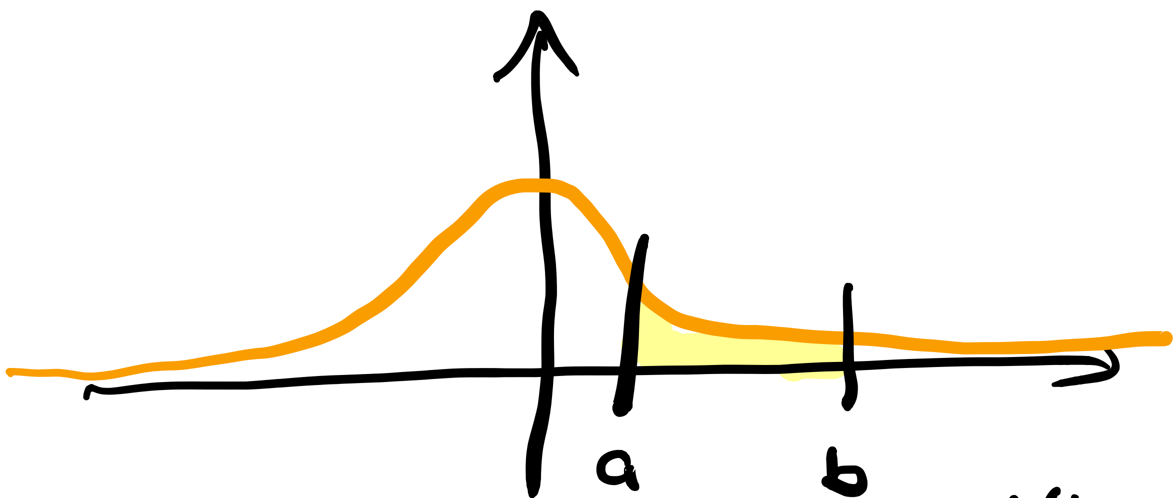
Examples:



Because $e^{-x^2/2}$ has no anti-derivative which uses known function one calls it

$$F(x) = \boxed{\mathrm{Erf}(x)}$$

F is useful because it allows to compute probabilities



What is the probability that the data are in the interval $[a,b]$

The answer is

$$P([a,b]) = \int_a^b f(x)\,dx$$

$$P((-\infty,\infty)) = \boxed{1}$$

Using F we can

express this as

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

$$\int_0^x f(t)\,dt = F(x) - F(0)$$

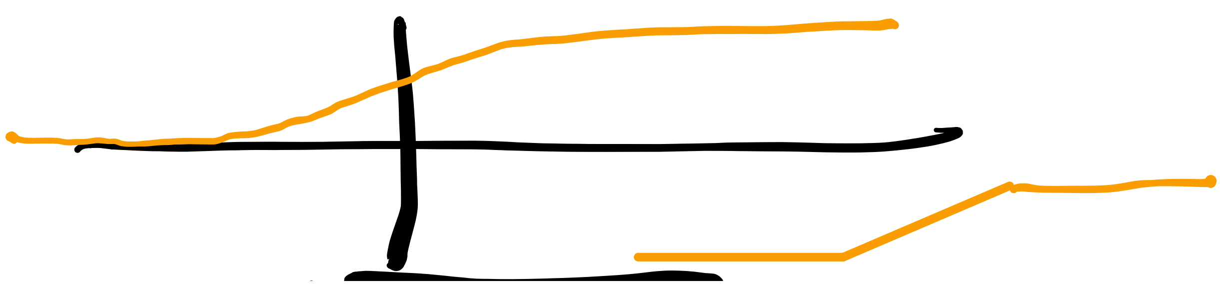
<u>FTC</u>

F is mononotinally increasing becaus

$F' = f \geq 0$

$F(-\infty) = 0$

$F(\infty) = 1$



4 $\boxed{\text{Mean}}$

If f is a PDF the <u>mean</u> m is defined as

$$\boxed{m = \int_{-\infty}^{\infty} x f(x)\,dx}$$

just for fun,

Variance is $\int_{-\infty}^{\infty} (x-m)^2 f(x)\,dx$

= Variance

$\sigma = \sqrt{\text{Variance}}$

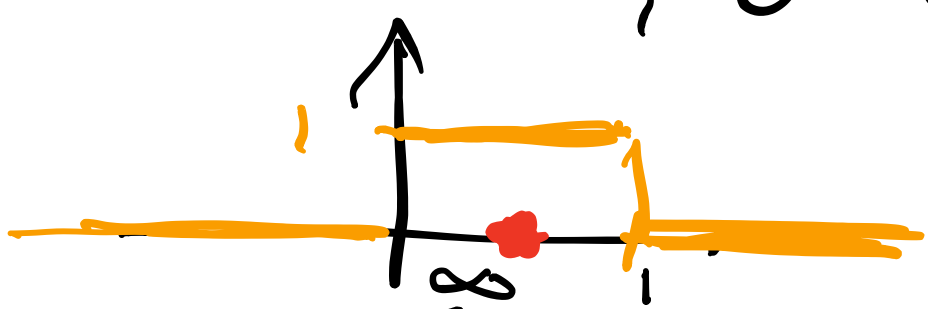
standard deviation

expected deviation from the mean.

m is called expectation

Examples!

a) $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$

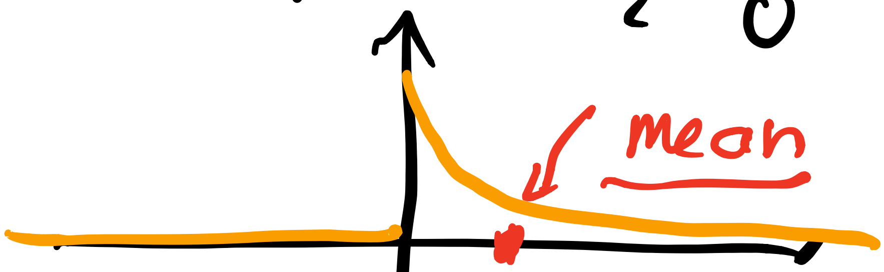


$$m = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_0^1 x \cdot 1 \cdot dx$$

$$= \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

b) $f(x) = \begin{cases} e^{-x}, & x \ge 0 \\ 0 & \text{else} \end{cases}$



What is m ?

$$\int_{-\infty}^{\infty} x e^{-x} \, dx = \int_{0}^{\infty} x e^{-x} \, dx$$

$$= e^{-x}(-1-x)\Big|_0^\infty$$

$$= -e^{-x} - x e^{-x}\Big|_0^\infty = \boxed{1}$$

$$= \lim_{t\to\infty} e^{-t}(-1-t) = 0 - \lim_{t\to\infty} \frac{t}{e^t}$$

l'Hopital : $\frac{1}{e^t} = 0$

Again!

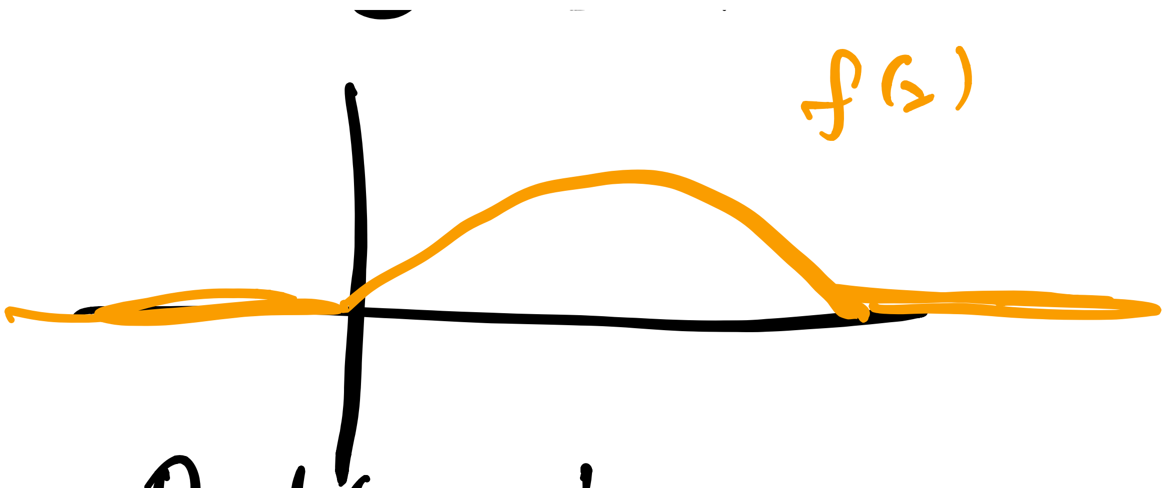
$$e^{-x}(-1-x)\Big|_0^\infty$$

If we plug in ∞,
then the limit is 0
using l'Hospital.

If we plug in 0
$e^{-0}(-1-0) = -1$
it is counter negative
(bottom part) $= \boxed{1}$

Breakout:

$f(x) = \sin(x)/2$
if $0 \leq x \leq \pi$
0 else



Problem!

$\boxed{\text{Check: } f \text{ is a PDF}}$