Unit 24

Substitution

Plan:

0) Remarks organ.
1) How it works.
2) Drill
3) Challenging probl

1) How does it work

$$\int x e^{-x^2/2}\,dx$$

$$= \boxed{-e^{-x^2/2} + C}$$

Check: Chain rule

$$\frac{d}{dx} - e^{-x^2/2}$$

$$= -e^{-x^2/2}\left(-\frac{2x}{2}\right)$$

✓

$$\int e^{-x^2/2}\, x \, dx$$

$$u = -\frac{x^2}{2}$$

$$du = -x \, dx$$

$$dx = \frac{-du}{x}$$

$$= \int e^{u} \cancel{x} \, \frac{-du}{\cancel{x}}$$

$$= -\int e^{u} \, du$$

$$= -e^{u} + C$$

$$\boxed{-e^{-x^2/2} + C}$$

A. Identify u
B. compute $du = u' \, dx$
C. Solve for dx
D. Substitute in formula
E. Integrate w.r.t u.
F. Back substitute.

Example:

$$\int e^{\sin(x)} \cos(x)\, dx$$

you can smell u!

$u = \sin x$
$du = \cos x\, dx$
$dx = \frac{du}{\cos x}$

$$= \int e^u \cancel{\cos x} \frac{du}{\cancel{\cos x}}$$

$$= \int e^u du$$

$$= e^u + C$$

$$= \boxed{e^{\sin x} + C}$$

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2 $\boxed{\text{DHII problems}}$

a) $\int \sqrt{3x+1}\, dx$

$$= \int \sqrt{u}\,\frac{du}{3}$$

$u = 3x+1$
$du = 3\,dx$
$dx = \frac{du}{3}$

$$= u^{3/2} \cdot \frac{2}{3} \cdot \frac{1}{3} + C$$

$$= u^{3/2}\frac{2}{9} + C$$

$$u = 3x+1$$
$$\frac{du}{dx} = 3$$
$$du = 3\,dx$$

$$= \boxed{(3x+1)^{3/2} \cdot \frac{2}{9} + C}$$

b) $\int \frac{\log(x+1)}{x+1}\,dx$

$$\left(= \int \frac{\log(u)}{u}\,du \right.$$

$u = x+1$
$du = dx$
$dx = du$

does not help get much $\Big)$

try again!

better:

$$\int \frac{\log(x+1)}{x+1}\,dx$$

$$=$$

$$u=\log(x+1)$$
$$du=\frac{1}{x+1}\,dx$$
$$dx=(x+1)\,du$$

$$=\int \frac{u}{\cancel{x+1}}\,\cancel{(x+1)}\,du$$

$$=\int u\,du$$

$$=\frac{u^2}{2}+C$$

$$=\boxed{\frac{\log^2(x+1)}{2}+C}$$

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c) $$\int \frac{\sin x}{\sqrt{\cos x}}\,dx$$

$$u=\cos x$$
$$du=-\sin x\,dx$$
$$dx=\frac{-du}{\sin x}$$

$$=\int \frac{\cancel{\sin x}}{\sqrt{u}}\,\frac{-du}{\cancel{\sin x}}$$

$$=\int \frac{(-1)}{\sqrt{u}}\,du$$

$$=-\int u^{-1/2}\,du=-u^{1/2}\cdot 2+C$$

$$=-\sqrt{u}\cdot 2+C$$

$$= \boxed{-\int \cos x \cdot 2 + C}$$

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d) $\int \tan(x)\,dx$

$$= \int \frac{\sin(x)}{\cos(x)}\,dx$$

$u = \cos x$

$du = -\sin x\,dx$

$dx = \frac{-du}{\sin x}$

$$= \int \frac{\cancel{\sin x}}{u} \frac{-du}{\cancel{\sin x}}$$

$$= \int \frac{-1}{u}\,du$$

$$= -\log(u) + C$$

$$= \boxed{-\log(\cos(x)) + C}$$

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why does it work?

$$\boxed{\int g'(\boxed{u(x)})\,u'(x)\,dx = g(u(x)) + C}$$

FTC + CHAIN:

$$\int \frac{d}{dx} g(u(x)) = \int g'(u(x))\, u'(x)$$

$$g(u(x))$$

3) $\boxed{\text{Tougher problems}}$

a) $$\int \frac{\sin x}{\cos^3 x}\, dx$$

$$= \int \frac{\cancel{\sin x}}{u^3} \frac{-du}{\cancel{\sin x}}$$

$u = \cos x$

$du = -\sin x\, dx$

$dx = \frac{-du}{\sin x}$

$$= \int \frac{-du}{u^3}$$

$$= \int -u^{-3} du$$

$$= \frac{-u^{-2}}{(-2)} + C$$

$$= \frac{u^{-2}}{2} + C = \boxed{\frac{1}{2\cos^2 x} + C}$$

$$\int \frac{\tan(x)}{\cos^2 x}\,dx$$

same integral!

$\frac{\sin x}{\cos x} = \tan x$

$=$

$u = \tan(x)$
$du = \frac{1}{\cos^2 x}\,dx$
$dx = \cos^2 x\,du$

$$\int \frac{u}{\cancel{\cos^2 x}} \cancel{\cos^2 x}\,du$$

$$= \int u\,du$$

$$= \frac{u^2}{2} + C$$

$$= \boxed{\frac{\tan^2(x)}{2} + C}$$

b) tough nut to crack!

$$\int \frac{x^3}{\sqrt{x^2+1}}\,dx$$

$=$

$$\int \frac{x^3}{\sqrt{u}} \frac{du}{2x}$$

$u = x^2+1$

$du = 2x\,dx$

$dx = \dfrac{du}{2x}$

$x^2 = u - 1$

$$= \int \frac{x^2}{\sqrt{u}\,2}\,du$$

$$= \int \frac{u-1}{\sqrt{u}\cdot 2}\,du$$

$$= \int \frac{u}{2\sqrt{u}} - \frac{1}{2\sqrt{u}}\,du$$

$$= \int \frac{\sqrt{u}}{2} - \frac{1}{2\sqrt{u}}\,du$$

$$= \int \frac{u^{1/2}}{2} - \frac{u^{-1/2}}{2}\,du$$

$$= \frac{u^{3/2}}{3} - u^{1/2} + C$$

$$= \boxed{\frac{(x^2+1)^{3/2}}{2} - (x^2+1)^{1/2} + C}$$

For definite

integrals.

$$\int_a^b f(x)\,dx$$

I recommend:

Compute F, get

$$\boxed{F(b) - F(a)}$$

The C does not matter.

c) tough but not too tough.

$$\int \frac{2x}{1+x^4}\,dx$$

$$= \int \frac{\cancel{2x}}{1+u^2}\,\frac{du}{\cancel{2x}}$$

$u = x^2$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$$= \int \frac{1}{1+u^2}\,du$$

$$= \arctan(u) + C$$

$$= \boxed{\arctan(x^2) + C}$$

Breakout:

A

why are the two answers

$$\boxed{\frac{1}{2\cos^2 x} + C}$$

$$\boxed{\frac{\tan^2 x}{2} + C}$$

the same ?

13 or solve

$$\boxed{\int \frac{1}{\log x \cdot x}\,dx}$$