Unit 28

Partial fractions

① Fractions

$$\frac{2}{3} + \frac{5}{7} = \frac{7\cdot 2 + 5\cdot 3}{21}$$

$$= \frac{29}{21}$$

<u>common denominator</u>

$$(3/3)/3 = 1/3 = \frac{1}{3}$$

$$3/(3/3) = 3 = 3$$

<u>PEMDAS wars</u>

<u>Pythagorean</u> : Music :

Harmonies

(2) $\boxed{\text{What do we know?}}$

a) $\displaystyle\int \frac{1}{x-7}\,dx = \log(x-7)+c$

b) $\displaystyle\int \frac{1}{x^2+1}\,dx = \arctan(x)+c$

c) $\displaystyle\int \frac{2x}{x^2+3}\,dx = \int \frac{1}{u}\,du$

$u = x^2+3$
$du = 2x\,dx$

$= \log(u)$
$= \log(x^2+3)$

We do not know how to integrate

$$\int \frac{1}{x^2-9}\,dx$$

Main idea:

Split

$$\boxed{\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}}$$

Find the constants A and B

$$\cancel{\frac{1}{x^2-9} \neq \frac{1}{x^2} - \frac{1}{9}}$$

Pitfall

$$\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab}$$

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3 $\boxed{\int \frac{1}{x^2-9}\,dx}$

$$\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}$$

$$= \frac{A(x+3)+B(x-3)}{x^2-9}$$

(common denominator)

we need

$$1 = A(x+3) + B(x-3)$$

$$1 = (A+B)x + 3A - 3B$$

linear equation

$$\boxed{\begin{aligned} 1 &= 3A - 3B \\ 0 &= A+B \end{aligned}}$$

$(x=0)$

$\frac{d}{dx}$

comparing coefficient

system of equations

$$B = -A$$

$$1 = 3A + 3A = 6A$$

$$\boxed{A = \frac{1}{6}}, \quad \boxed{B = -\frac{1}{6}}$$

Now we know :

$$\int \frac{1}{x^2-9}\,dx = \int \frac{1/6\,dx}{x-3} - \int \frac{1/6}{x+3}\,dx$$

Solved the integral $= \boxed{\frac{1}{6}\log(x-3) - \frac{1}{6}\log(x+3) + C}$

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④ $\boxed{\text{A magic way}}$

$$\frac{1}{x^2-9} (x-3) = \frac{A}{x-3}(x-3) + \frac{B}{x+3}(x-3)$$

$$\frac{1}{x+3} = A + \frac{B(x-3)}{x+3}$$

Now set $x = 3$!

$$\boxed{\frac{1}{6} = A} + 0$$

$$\frac{1\,(x+3)}{x^2-9} = \frac{A\,(x+3)}{x-3} + \frac{B\,\cancel{(x+3)}}{\cancel{x+3}}$$

$$\frac{1}{x-3} = \frac{A\,(x+3)}{x-3} + B$$

Now set $x = -3$

$$\boxed{-\frac{1}{6} = B}$$

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5 **Complicated**

$$\frac{1}{(x-1)(x-2)(x+1)}$$

$$\frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x+1}$$

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common denomidor

(complicated way)

$$1 = A(x-2)(x+1) + B(x-1)(x+1) + C(x-1)(x-2)$$

FOIL

$$1 = (A+B+C)x^2 + (-A+0B-3C)x - 2A - 1B + 2C$$

$$\boxed{\begin{aligned} 1 &= -2A - B + 2C \\ 0 &= -A + 0B - 3C \\ 0 &= A + B + C \end{aligned}}$$

System of equation

we do not solve.

Lets look at the magic method.

$$\frac{1}{\cancel{(x-1)}(x-2)(x+1)}\cancel{(x-1)} = \frac{A\cancel{(x-1)}}{\cancel{x-1}} + \frac{B(x-1)}{x-2} + \frac{C(x-1)}{x+1}$$

and put $x = 1$

$$\frac{1}{(1-2)(1+1)} = A + B \cdot 0 + C \cdot 0$$

$$A = -\frac{1}{2}$$

$$\frac{1}{(x-1)(x-2)(x+1)}(x-2) = \frac{A(x-2)}{x-1} + \frac{B(x-2)}{x-2} + \frac{C(x-2)}{x+1}$$

and put $x = 2$

$$\frac{1}{(2-1)(2+1)} = 0 + B + 0$$

$$B = \frac{1}{3}$$

$$\frac{1}{(x-1)(x-2)(x+1)}(x+1) = \frac{A(x+1)}{x-1} + \frac{B(x+1)}{x-2} + \frac{C(x+1)}{x+1}$$

and put $x = -1$

$$\frac{1}{(-1-1)(-1-2)} = 0 + 0 + C$$

$$C = \boxed{\frac{1}{6}}$$

Solve the integral

$$\int \frac{1}{(x-1)(x+1)(x-2)}\,dx$$

$$= \int \frac{-1/2}{x-1}\,dx$$

$$+ \int \frac{1/3}{x-2}\,dx$$

$$+ \int \frac{1/6}{x+1}\,dx$$

$$\boxed{= \frac{-1}{2}\log(x-1) + \frac{1}{3}\log(x-2) + 1/6\log(x+1) + c}$$

6 **What can we do**

a) <u>Example</u>: $\dfrac{(3x+1)\,\cancel{(x-1)}}{\cancel{(x-1)}(x+1)(x-2)}$ $\quad \dfrac{4}{-2} = -2$

Do the same thing:

$$\frac{A\,(x-1)}{x-1} + \cancel{\frac{B\,(x-1)}{x+1}} + \cancel{\frac{C\,(x-1)}{x-2}}$$

A

and put $x=1$

To get the coefficients just do the same.

$A=-2$

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b) <u>Example</u>: $\dfrac{1}{(x-1)^2(x+2)}$

$$= \frac{A}{x-1} + \frac{B}{x-1} + \frac{C}{x+2}$$

does not work this

does not work

works

$$= \frac{A}{x-1} + \frac{Bx}{x-1} + \frac{C}{x+2}$$

Now we can find the coefficients.

Then we have to integrate $\frac{x}{x-1}$.

How do we do that?

$$\int \frac{x}{x-1}\,dx = \int \frac{u+1}{u}\,du$$

$u = x-1$

$du = dx$

$x = u+1$

$$\int 1 + \frac{1}{u}\,du$$

$$= u + \log u + c$$

$$= (x-1) + \log(x-1) + c$$

c) $\int \frac{x^2+1}{x^2-1}\,dx$

$$\frac{x^2+1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

<u>does not work.</u>

How can we get rid of the x^2

$$\frac{x^2+1}{x^2-1} - \frac{x^2-1}{x^2-1} = \frac{2}{x^2-1}$$

$$\frac{x^2+1}{x^2-1} = \frac{2}{x^2-1} + 1$$

this we solve using $\frac{A}{x-1} + \frac{B}{x+1}$

To factor x^2+1

$$=(x-i)(x+i)$$

Using complex numbers

$$\frac{1}{x^2+1} = \frac{A}{x-i} + \frac{B}{x+i}$$

arctan(x)

$A\log(\cancel{x}-i)$
$+ B\log(x+i)$

$i^2=-1$

With this method you can solve in principle any

$$\int \frac{p(x)}{q(x)}\,dx$$

where p and q are polynomials.

Breakout:

a) $$\int \frac{1}{(x-1)(x+5)}\,dx$$

$$\frac{A}{(x-1)} + \frac{B}{x+5}$$

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b) $\displaystyle\int \frac{1}{x^2+2x+1}\,dx$

$$= \int \frac{1}{(x+1)^2}\,dx$$

(Partial fractions!

$$\frac{1}{(x+1)^2} = \frac{A}{(x+1)} + \frac{Bx}{(x+1)}$$

$(x+1) \quad (x+1) \quad (x+1)$

Simpler way:

Substitution

$u = x+1$
$du = dx$

$$= \int \frac{1}{u^2}\,du$$

$$= -\frac{1}{u} + C$$

$$= \boxed{-\frac{1}{(x+1)} + C}$$

General advise:

- First try substitution
- Then try parts
- Try partial fraction

Next time:

trig substitution