Unit 29

Trig Substitution

① Square roots
② Circle area
③ A general method
trig rational function
④ Cheese problem

① $\int \frac{1}{\sqrt{1-x^2}}\,dx$

$x = \sin u$
$dx = \cos u\, du$
$\sqrt{1-x^2} = \sqrt{1-\sin^2 x} = \cos x$

trig substit.

$= \int \frac{\cos u\, du}{\cos u} = \int 1\, du$

$= u + C = \boxed{\arcsin x + C}$

2) $\underline{\int \sqrt{1-x^2}\,dx}$ $=$ x

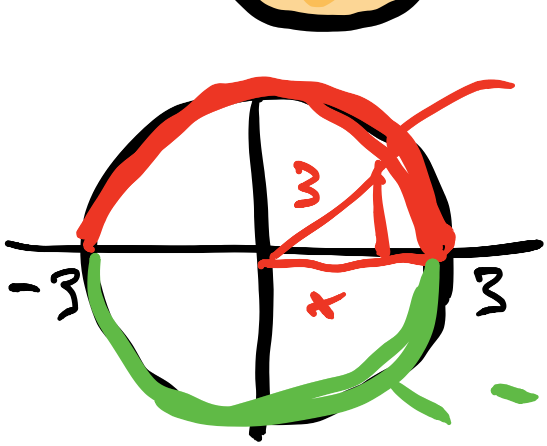
$$\int \cos^2 u \, du$$

$$= \int \frac{1+\cos 2u}{2}\, du$$

$$= \frac{u}{2} + \frac{\sin 2u}{4} + C$$

$$= \boxed{\frac{\arcsin(x)}{2} + \frac{\sin(2\arcsin x)}{4} + C}$$

This integral appeared in 3 computation



$\sqrt{9-x^2}$ Pythagora

$-\sqrt{9-x^2}$

3

$$A = \int_{-3}^{3} 2\sqrt{9-x^2}\, dx$$

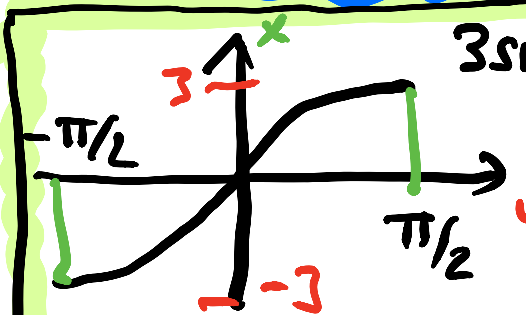
$$= \int_{-\pi/2}^{\pi/2} 2\cdot 9 \cos^2(u)\, du$$

$x = 3\sin u$

$dx = 3\cos u\, du$

$\sqrt{9-x^2} = 3\cos u$

Substitute the bounds too.



$\pi/2$

x goes from −3 to 3

u goes from

$$18\int_{-\pi/2}^{\pi/2} \cos^2 u\, du \qquad \frac{1+\cos 2u}{2}$$

$$= 18\left(\frac{u}{2} + \frac{\sin 2u}{4}\right)\Bigg|_{-\pi/2}^{\pi/2}$$

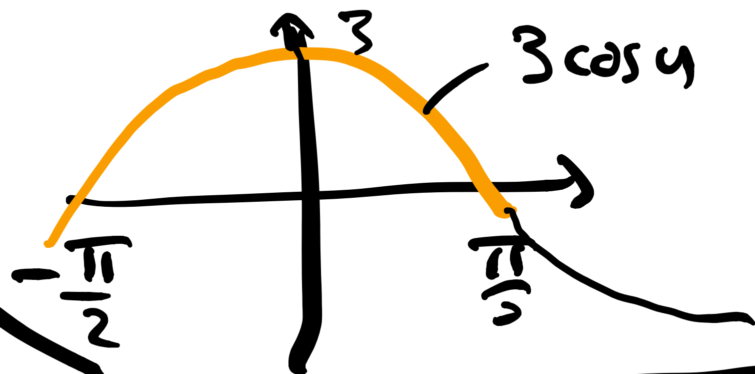
$$= 18\,\frac{\pi}{2}$$

$$= \boxed{9\pi}$$

$r^2\pi$

Area

(If we would have chosen $x = 3\cos u$



)

---

③ $\boxed{\text{General magic}}$

$$\int \frac{\cos x + \sin x}{3\cos x + 2\sin^2 x}\, dx$$

can be integrated with a general substit procedure

Here is the magic!

$$x = \tan\left(\frac{u}{2}\right)$$

$$dx = \frac{2\,du}{1+u^2}$$

$u = 2\arctan x$

$$\sin x = \frac{2u}{1+u^2}$$

$$\cos x = \frac{1-u^2}{1+u^2}$$

To prove this, just check and use identi'l. like

$$\sin x = 2\cos\frac{x}{2}\sin\frac{x}{2}$$

$$= 2\tan\frac{x}{2}\cdot\cos^2\frac{x}{2}$$

$$\frac{1}{\cos^2\frac{x}{2}} = 1+\tan^2\frac{x}{2}$$

...

Lets solve some integrals

a) $\displaystyle\int \frac{1}{\cos x}\, dx =$

$$\int \frac{\cancel{1+u^2}}{1-u^2}\,\frac{2du}{\cancel{1+u^2}}$$

$$= \int \frac{2}{1-u^2}\, dx$$

How do we do this?

Partial fractions!

$$\frac{A}{1-u} + \frac{B}{1+u} \cdots$$

$$= \int \frac{1}{1-u} + \frac{1}{1+u} \, du$$

$$= -\log(1-u) + \log(1+u) + C$$

$$= \boxed{-\log(1-2\arctan x) + \log(1+2\arctan x) + C}$$

b) why not try your self on a piece of paper

$$\int \frac{1}{\sin x}\, dx$$

$$= ?$$

$$\sin x = \frac{2u}{1+u^2}$$

$$dx = \frac{2\, du}{1+u^2}$$

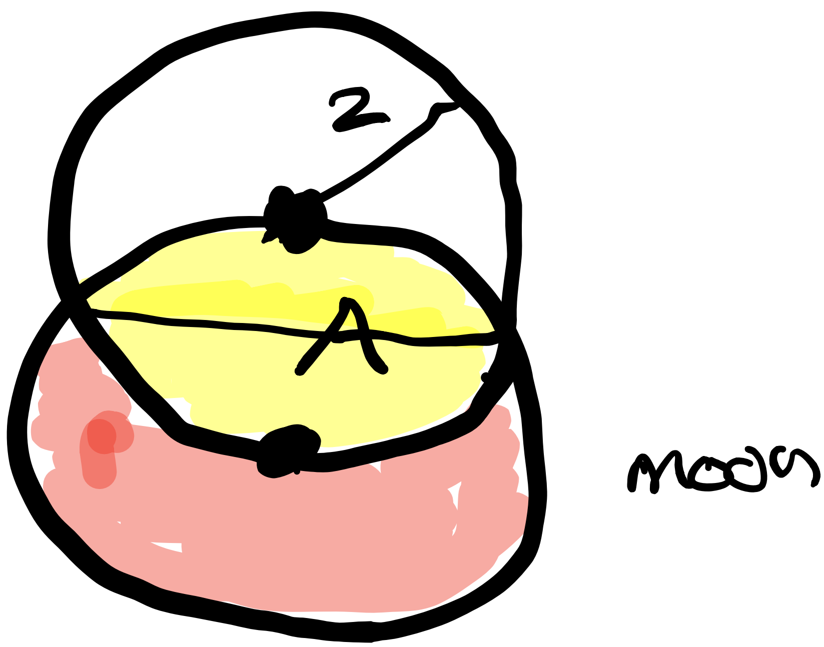
$$= \int \frac{1}{u}\, du$$

$$= \log u + c$$

$$= \boxed{\log(2\arctan x) + c}$$

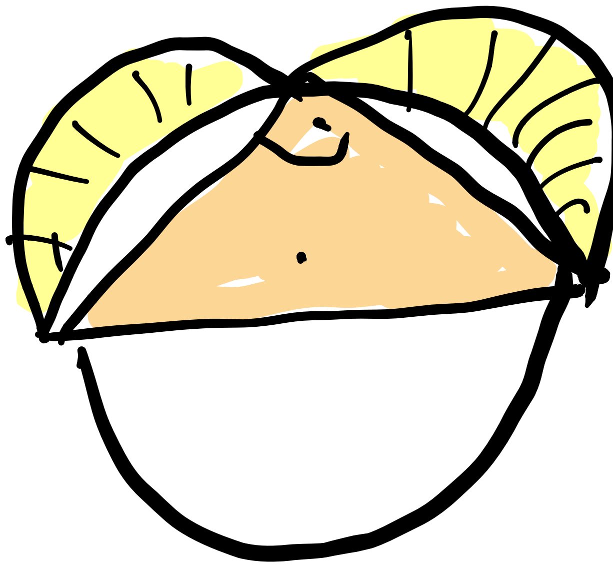
④ **Raclette**

<u>Cheese meal</u>



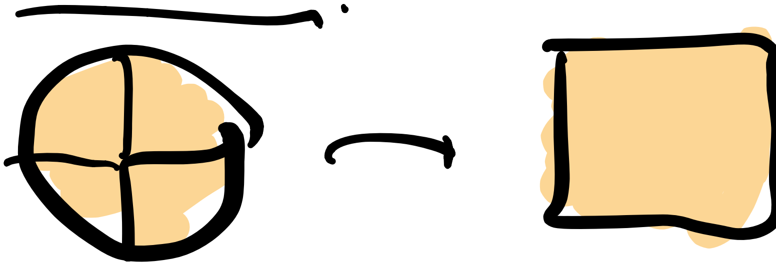
Find the Area A.

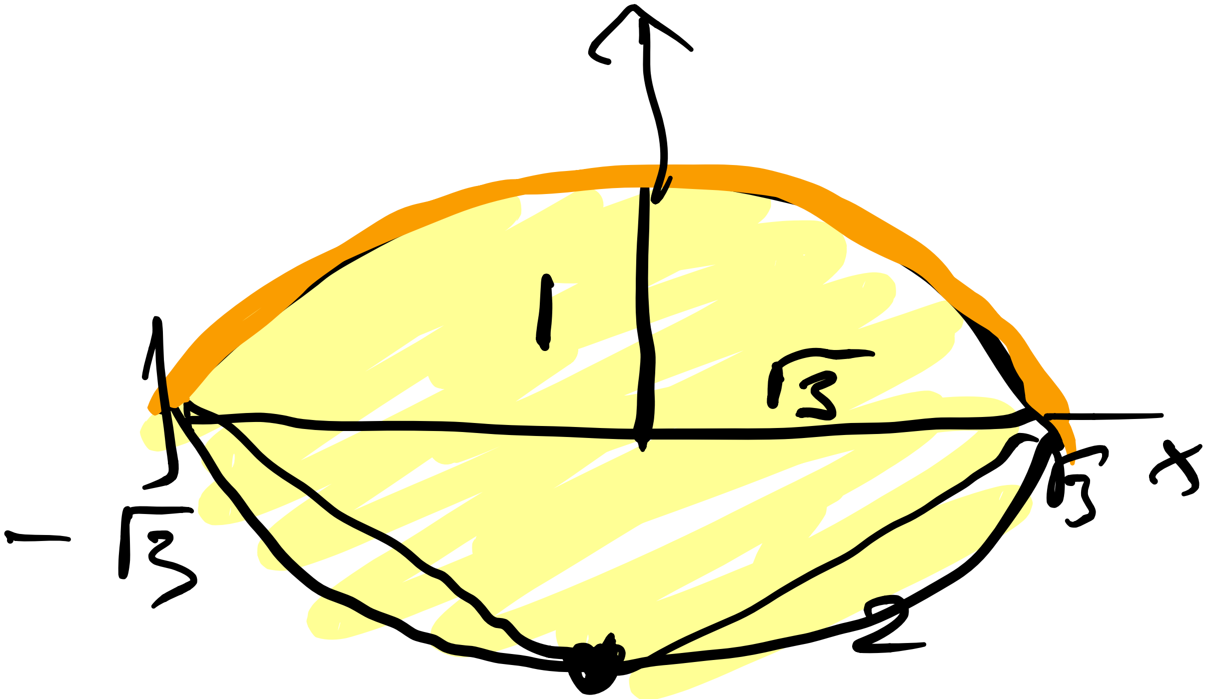
Greeks:



Area of moons = Area of triangle.

Holy grail was quadrature of the circle!





$$x^2+(y+1)^2=4$$

$$y=\sqrt{4-x^2}-1$$

is graph 

$$2\int^{\sqrt{3}} \sqrt{4-x^2}-\boxed{1}\,dx$$

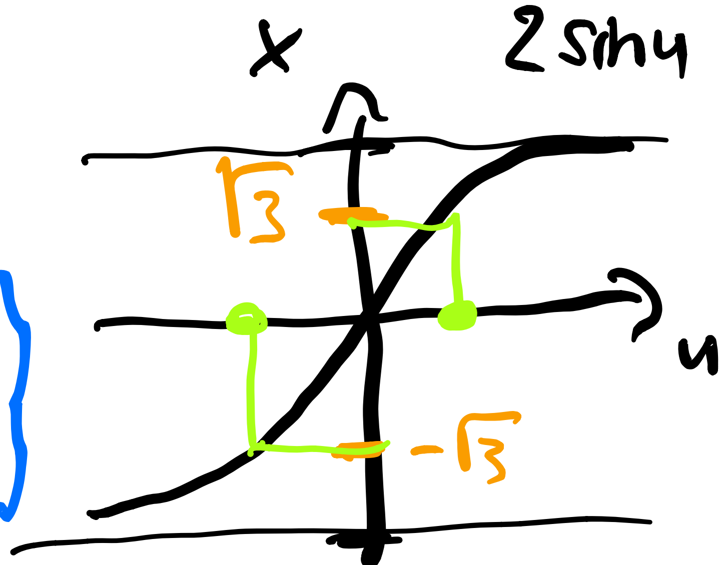
$-\sqrt{3}$ $\boxed{2\int_{-\sqrt{3}} 1\,dx = 4\sqrt{3}}$ $= A$

$=$

$x = 2\sin u$

$dx = 2\cos u\,du$

$\sqrt{4-x^2} = 2\cos u$



$\sqrt{4-x^2} = \sqrt{4-4\sin^2 u} = 2\sqrt{1-\sin^2 u} = 2\cos u$

$2\cdot\sin u = \sqrt{3}$

$\sin u = \frac{\sqrt{3}}{2}$

$u = \frac{\pi}{3}$

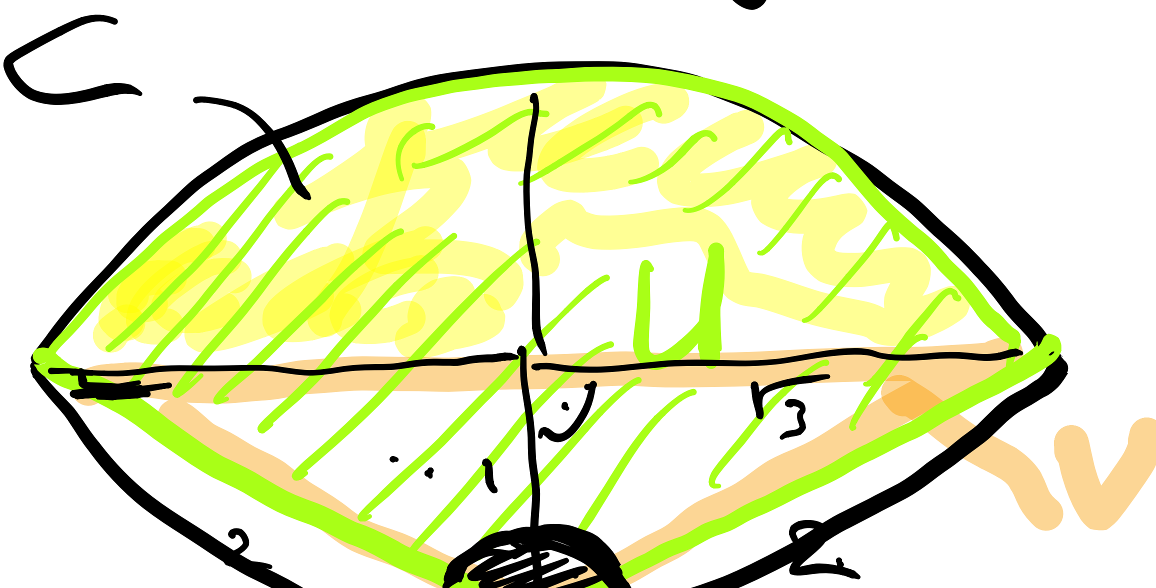
$$= 2\int_{\boxed{-\pi/3}}^{\boxed{\pi/3}} 4\cos^2 u\,du - \boxed{4\sqrt{3}}$$

$$8\int_{-\pi/3}^{\pi/2} \frac{(1+\cos 2u)}{2}\,du$$

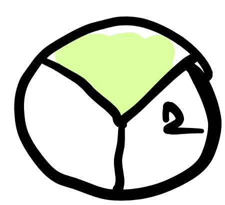
$$= \frac{8\pi}{3} + 2\sqrt{3} - 4\sqrt{3}$$

$$= \boxed{\frac{8\pi}{3} - 2\sqrt{3}}$$

Elementary way!



120°



$$U = 4\pi/3$$

$V = \sqrt{3}$ triangle area

C cheese area

$$\boxed{C = U - V}$$

$$A = 2C = 2U - 2V$$

$$= \boxed{\frac{8\pi}{3} - 2\sqrt{3}}$$