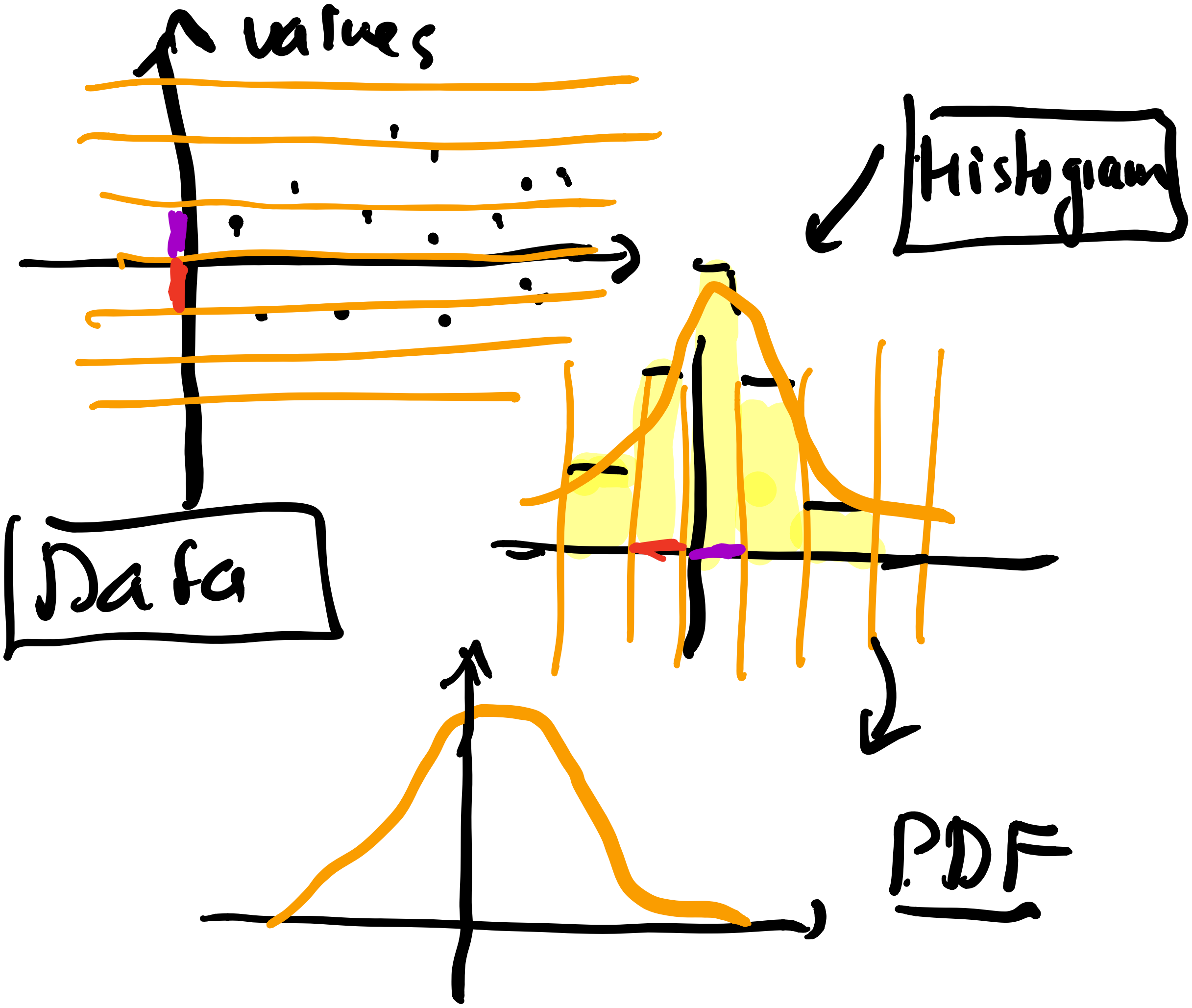
Unit 20

Calculus and Stats

(1) Data → Functions



Presentation Animations

2 $\boxed{\text{Moments}}$ $f =$ PDF

$$M_n = \int_{-\infty}^{\infty} x^n f(x)\, dx$$

n'th moment of f.

$$M_0 = \int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$M_1 = \int_{-\infty}^{\infty} x f(x)\, dx = m$$

mean

average expectation

$$M_2 = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

Variance

$$\mathrm{Var}(f) = \boxed{M_2 - m^2}$$

$$= \int_{-\infty}^{\infty} (x-m)^2 f(x)\, dx$$

$$\int_{-\infty} (x-m)^2 f(x)\,dx$$

Why is the same?

FOIL: $\displaystyle\int_{-\infty}^{\infty}(x^2-2mx+m^2)f(x)\,dx$

$$= M_2 - 2m M_1 + m^2 \cdot 1$$

$$= M_2 - 2m^2 + m^2$$

$$= M_2 - m^2$$

$\sigma(f) = \sqrt{Var(f)}$ <u>Standard deviation</u>

expected deviation from the mean.

M_3 comes in for <u>Skewness</u>

$$\frac{\displaystyle\int_{-\infty}^{\infty}(x-m)^3 f(x)\,dx}{\sigma^3}$$

Centered moment

$$C_n = \int_{-\infty}^{\infty} (x-m)^n f(x)\,dx$$

Centered normalized mo

$$\frac{C_n}{\sigma_n} \qquad \frac{C_3}{\sigma_3}$$ Skewness

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③ Compute some examples

a) $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$\sqrt{2\pi}\, M_2 = \int_{-\infty}^{\infty} x^2 e^{-x^2/2}\, dx$$

What technique do we use: parts

naive $u = x^2$, $dv = e^{-x^2/2}$

does not work.

we have to be more clever

$$\int_{-\infty}^{\infty} x \cdot x e^{-x^2/2}\, dx$$

$$= -x e^{-x^2/2} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} 1 \cdot e^{-x^2/2}\, dx$$

$$= 0 + \int_{-\infty}^{\infty} e^{-x^2/2}\, dx$$

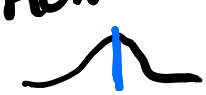
$=$ "Gifted" $\sqrt{2\pi}$

$$\boxed{m = 0}$$

$$\sqrt{2\pi}\, M_2 = \sqrt{2\pi}$$

$$M_2 = 1$$

see homework



$m=0$

$$Var(f) = M_2 - m^2$$
$$= 1$$
$$\sigma(f) = \sqrt{1} = 1$$

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b) $\boxed{M_2 \text{ for exp}}$

$$M_2 = \int_0^\infty x^2 e^{-x}\,dx$$

What technique?

Typical application Tic tac toe

x^2	e^{-x}	signs
$2x$	$-e^{-x}$	1
2	e^{-x}	-1
0	$-e^{-x}$	1

$$-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\Big|_0^\infty$$
$$= 2$$

$M_2 = 2$

$M_1 = 1$ mean

$Var(f) = M_2 - M_1^2$

$= 2 - 1 = \boxed{1}$

$\sigma(f) = \sqrt{Var f} = \boxed{1}$

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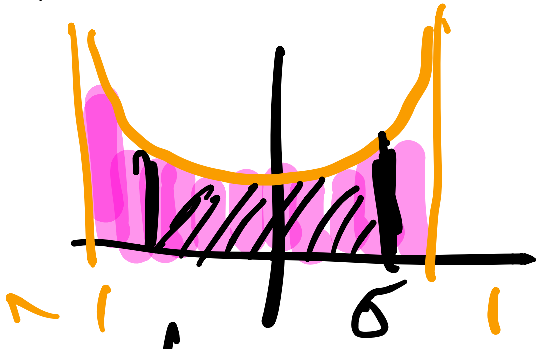
c) $\boxed{\text{Var of arcsin}}$



$$f(x) = \frac{1}{\pi\sqrt{1-x^2}}$$

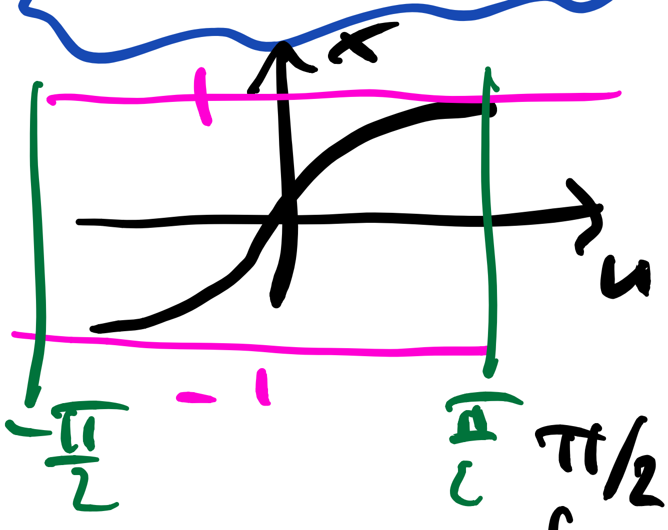
$$\frac{1}{\pi}\int_{-1}^{1}\frac{x^2}{\sqrt{1-x^2}}\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\sin^2 u\,du$$

$x = \sin u$

$dx = \cos u\,du$

$\sqrt{1-x^2} = \cos u$

$$\frac{dx}{\sqrt{1-x^2}} = du$$



$$= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{(1-\cos 2u)}{2}\,du$$

$$-\pi/2$$

$$= \frac{1}{\pi}\left(\frac{u}{2} - \frac{\sin 2u}{4}\right)\Bigg|_{-\pi/2}^{\pi/2}$$

$$= \frac{\pi}{2\pi} = \boxed{\frac{1}{2}} = M_2$$

$$\mathrm{Var}(f) = M_2 - M_1^2$$

$$= M_2 = \frac{1}{2}$$

$$\sigma(f) = \boxed{\sqrt{\frac{1}{2}}}$$

$$0.7$$