

## 5/7/2020: Final Practice A

Your Name:

- Solutions are submitted as PDF handwritten in a file called after your name. Capitalize the first letters like OliverKnill.pdf. The paper has to **feature your personal handwriting** and contain no typed part. If you like, you can start writing on a new paper. For 1), you could write 1: False, 2: False  $\dots$  20: False. Sign your paper.
- No books, calculators, computers, or other electronic aids are allowed. You can use one page of your own handwritten notes when writing the paper. It is your responsibility to submit the paper on time and get within that time also a confirmation.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

Problem 1) TF questions (20 points). No justifications are needed.

1)  T  F  $\cos(17\pi/4) = \sqrt{2}/2.$

**Solution:**

Yes, it is  $\cos(\pi/4)$ .

2)  T  F The tangent function is monotonically increasing on the open interval  $(-\pi/2, \pi/2)$ .

**Solution:**

Indeed, its derivative is  $1/\cos^2(x)$ .

3)  T  F The arccot function is monotonically increasing from  $\pi/4$  to  $3\pi/4$ .

**Solution:**

It is decreasing.

4)  T  F If  $f$  is a probability density function, then  $\int_{-\infty}^{\infty} f(x) dx = 0$

**Solution:**

It is 1.

5)  T  F  $\frac{d}{dx} e^{\log(x)} = 1.$

**Solution:**

First simplify.

6)  T  F If  $f''(0) = -1$  then  $f$  has a local maximum at  $x = 0$ .

**Solution:**

We need the first derivative to be zero

- 7)  T  F The improper integral  $\int_{-1}^1 1/|x| dx$  is finite.

**Solution:**

This is an improper integral which does not exist

- 8)  T  F The function  $-\cos(x) - x$  has a root in the interval  $(-100, 100)$ .

**Solution:**

Use the intermediate value theorem.

- 9)  T  F If a function  $f$  has a local maximum in  $(0, 1)$  then it also has a local minimum in  $(0, 1)$ .

**Solution:**

Take  $f(x) = -x^2$ . It does not have a local minimum.

- 10)  T  F The anti derivative of  $1/(1 - x^2)$  is equal to  $\arctan(x)$ .

**Solution:**

It is an other sign.

- 11)  T  F The function  $f(x) = (e^x - e^{2x})/(x - x^2)$  has the limit 1 as  $x$  goes to zero.

**Solution:**

Use Hospital's rule to see that it is the same than the limit  $(e^x - 2e^{2x})/(1 - 2x)$  for  $x \rightarrow 0$  but this is  $-1$ .

- 12)  T  F If you listen to the sound  $e^{-x} \sin(10000x)$ , then it gets louder and louder as time goes on.

**Solution:**

The amplitude decays like  $e^{-x}$ .

- 13)  T  F The function  $f(x) = e^{x^2}$  has a local minimum at  $x = 0$

**Solution:**

The function is positive near 0 but equal to zero at 0.

- 14)  T  F The function  $f(x) = (x^{55} - 1)/(x - 1)$  has the limit 1 for  $x \rightarrow 1$ .

**Solution:**

Use Hopital's rule, or heal the function. The limit is 55.

- 15)  T  F If the total cost  $F(x)$  of an entity is extremal at  $x$ , then we have a break even point  $f(x) = g(x)$ .

**Solution:**

This is not the strawberry theorem.

- 16)  T  F The value  $\int_{-\infty}^{\infty} xf(x) dx$  is called the expectation of the PDF  $f$ .

**Solution:**

Yes this is true

- 17)  T  F The trapezoid rule is an integration method in which the left and right Riemann sum are averaged.

**Solution:**

This is a good description

18)  T  F  $\tan(\pi/3) = \sqrt{3}$ .

**Solution:**

Yes, it is equal to  $\sin(\pi/6)$ .

19)  T  F A Newton step for the function  $f$  is  $T(x) = x + \frac{f(x)}{f'(x)}$ .

**Solution:**

Wrong. The sign is off.

20)  T  F  $\sin(\arctan(1)) = \sqrt{3}$ .

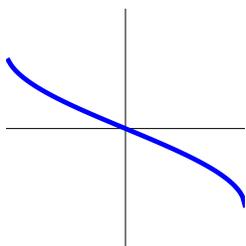
**Solution:**

We have  $\arctan(1) = \pi/4$  and so  $\sin(\arctan(1)) = \sqrt{3}$ .

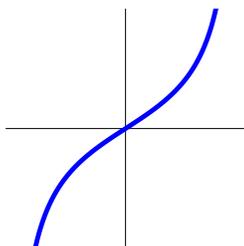
Problem 2) Matching problem (10 points) No justifications needed

(5 points) Match the functions names with their graphs (1-4) their derivatives (A-D) (middle row) and second derivatives (a-d) (last row).

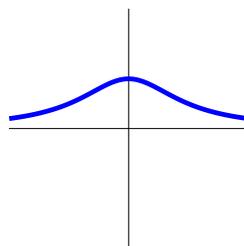
Function	fill in 1)-4)	fill in A)-D)	fill in a)-d)
$\sin(x)/x$			
$\tan(x)$			
$\arcsin(x)$			
$1/(1+x^2)$			



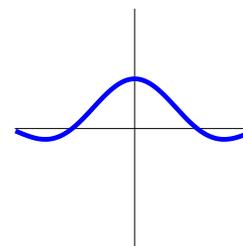
1)



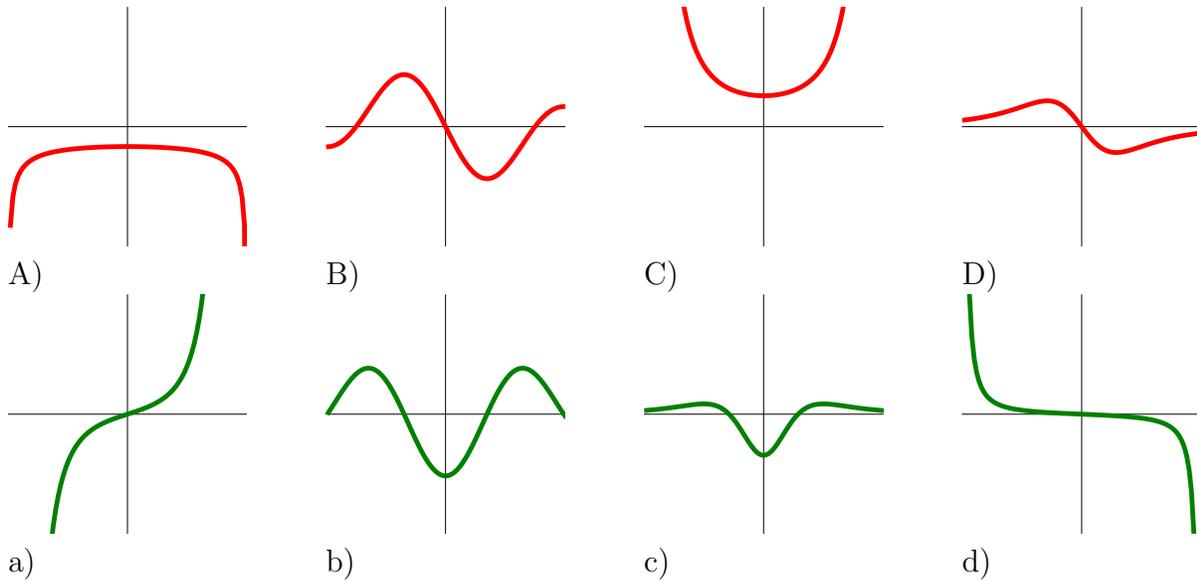
2)



3)



4)



(5 points) Which of the following limits exists in the limit  $x \rightarrow 0$ .

Function	exists	does not exist
$\sin^4(x)/x^4$		
$1/\log x $		
$\arctan(x)/x$		
$\log x /(x-1)$		
$\cos(x)/(x-1)$		
$(x^{10}-1)/(x-1)$		

**Solution:**

a) 4,B,b

2,C,a

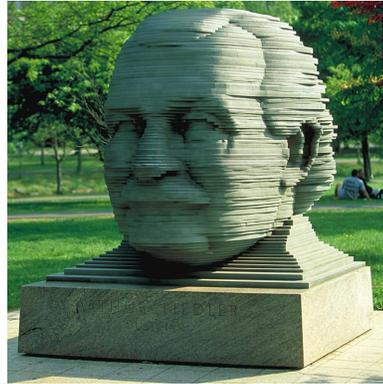
1,A,d

3,D,c

b) Every limit exists except the case  $\log|x|/(x-1)$ .

Problem 3) Matching or short answer problem (10 points). No justifications are needed.

a) (4 points) On the Boston Esplanade is a sculpture of **Arthur Fiedler** (1894-1979) a long-time conductor of the Boston Pops Orchestra. His head is sliced into thin slices. Assume that the thickness of each level is  $h = 1.5$  inch and the area of each of the 100 slices  $k$  is  $A(k)$ . Which formula gives the volume of the head? (One applies.)



Formula	Check if true
$1.5[A(1) + \dots + A(100)]$	<input type="checkbox"/>
$\frac{1}{1.5}[A(1) + \dots + A(100)]$	<input type="checkbox"/>

Formula	Check if true
$1.5[\frac{1}{A(1)} + \dots + \frac{1}{A(100)}]$	<input type="checkbox"/>
$\frac{1.5}{100}[A(1) + \dots + A(100)]$	<input type="checkbox"/>

b) (4 points) The summer has arrived on May 12 2014 for a day before it cooled down again. Harvard students enjoy the **Lampoon pool** that day in front of the **Lampoon castle**. Assume the water volume at height  $z$  is  $V(z) = 1 + 5z - \cos(z)$ . Assume water evaporates at a rate of  $V'(z) = -1$  gallon per day. How fast does the water level drop at  $z = \pi/2$  meters? Check the right answer: (one applies)



Rate	Check if true
-6	<input type="checkbox"/>
-1/6	<input type="checkbox"/>

Rate	Check if true
-4	<input type="checkbox"/>
-1/4	<input type="checkbox"/>

c) (2 points) Speaking of weather: the temperature on May 13, 2014 in Cambridge was 52 degrees Fahrenheit. The day before, on May 12, the temperature had been 85 degrees at some point and had us all dream about **beach time**. Which of the following theorems assures that there was a moment during the night of May 12 to May 13 that the temperature was exactly 70 degrees? (One applies.)



Theorem	check if true
Mean value theorem	<input type="checkbox"/>
Rolle theorem	<input type="checkbox"/>

Theorem	check if true
Intermediate value theorem	<input type="checkbox"/>
Bolzano theorem	<input type="checkbox"/>

**Solution:**

$1.5[A(1) + \dots + A(100)]$  is the Riemann sum because  $dz = 1.5$ .

b)  $z' = -1/6$  as

$$V' = 5z' + \sin(z)z' = -1$$

gives for  $z = \pi/2$  the equation  $6z' = -1$  leading to the result. The other selections could made sense if some mistake was done like writing  $5 + \sin(z)z' = -1$  for example which would lead to  $z' = -6$  which is false.

c) It is the intermediate value theorem.

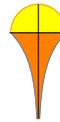
Problem 4) Area computation (10 points)

Find the area enclosed by the graphs of the functions

$$f(x) = \log|x|$$

and

$$g(x) = \sqrt{1-x^2}.$$



**Solution:**

The integral is

$$\int_{-1}^1 \sqrt{1-x^2} - \log|x| dx.$$

The first integral is  $\pi/2$  as it is the area of half the circle. The second integral is  $-2 \int_0^1 \log|x| dx = 2(x - x \log(x))|_{-1}^1 = 4$ . The answer is  $\pi/2 + 4$ .

Problem 5) Volume computation (10 points)

The lamps near the front entrance of the **Harvard Malkin Athletic Center** (MAC) have octagonal cross sections, where at height  $z$ , the area is

$$A(z) = 2(1 + \sqrt{2})(1 + z)^2$$

with  $0 \leq z \leq 3$ . What is the volume of the lamp?



**Solution:**

$$(2 + 2\sqrt{2}) \int_0^3 (1 + z)^2 dz = 21(2 + \sqrt{8}) = 42 + 21\sqrt{8} = \boxed{42(1 + \sqrt{2})}.$$

Problem 6) Improper integrals (10 points)

Which of the following limits  $R \rightarrow \infty$  exist? If the limit exist, compute it.

a) (2 points)  $\int_1^R \sin(2\pi x) dx$

b) (2 points)  $\int_1^R \frac{1}{x^2} dx$

c) (2 points)  $\int_1^R \frac{1}{\sqrt{x}} dx$

d) (2 points)  $\int_1^R \frac{1}{1+x^2} dx$

e) (2 points)  $\int_1^R x dx$

**Solution:**

a)  $\lim_{R \rightarrow \infty} -\cos(2\pi x)|_1^R$  does not exist.

b)  $\lim_{R \rightarrow \infty} -\frac{1}{x}|_1^R = 1$  exists.

c)  $\lim_{R \rightarrow \infty} 2\sqrt{x}|_1^R$  does not exist.

d)  $\lim_{R \rightarrow \infty} \arctan(x)|_1^R = \pi/2 - \pi/4 = \pi/4$ .

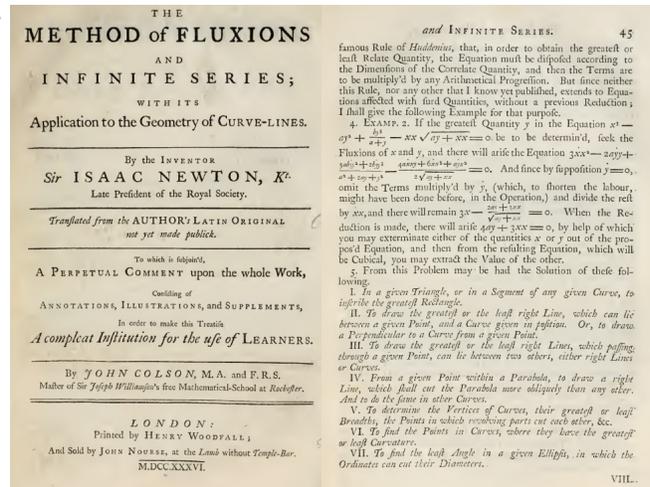
e)  $\lim_{R \rightarrow \infty} R^2/2$  does not exist.

Problem 7) Extrema (10 points)

In Newton’s masterpiece ”The Method of Fluxions” on the bottom of page 45, Newton asks: ”In a given triangle or in a segment of any given curve, to inscribe the greatest rectangle.” Lets be more specific and find rectangle with largest area

$$A = xy$$

in the triangle given by the x-axes, y-axes and line  $y = 2 - 2x$ . Use the second derivative test to make sure you have found the maximum.



**Solution:**

The function to extremize is  $f(x) = x(2 - 2x) = 2x - 2x^2$ . Its derivative is  $f'(x) = 2 - 4x$ . It is zero if  $x = 1/2$ . The second derivative is  $f''(x) = -4$ . As it is negative, the extremum is a **maximum**.

Problem 8) Integration by parts (10 points)

a) (5 points) Find

$$\int (1 + x + x^2 + x^3 + x^4)(\sin(x) + e^x) dx .$$

b) (5 points) Find

$$\int \log(x) \frac{1}{x^2} dx .$$

**Solution:**

a) Use TicTacToe:

$1 + x + x^2 + x^3 + x^4$	$\sin(x) + e^x$	
$1 + 2x + 3x^2 + 4x^3$	$-\cos(x) + e^x$	$\oplus$
$2 + 6x + 12x^2$	$-\sin(x) + e^x$	$\ominus$
$6 + 24x$	$\cos(x) + e^x$	$\oplus$
$24$	$\sin(x) + e^x$	$\ominus$
$0$	$-\cos(x) + e^x$	$\oplus$

Collecting together, we could write  $(4x^3 + 3x^2 - 22x - 5) \sin(x) + (x^4 - 3x^3 + 10x^2 - 19x + 20)e^x + (-x^4 - x^3 + 11x^2 + 5x - 23) \cos(x)$ .

b) As we know from LIATE, we differentiate the log  $\log(x)$ . We get

$$-\log(x) \frac{1}{x} + \int \frac{1}{x^2} = -\log(x)/x - 1/x + C .$$

Problem 9) Substitution (10 points)

a) (5 points) **“One,Two,Three,Four Five, once I caught a fish alive!”**

$$\int \frac{(1 + 2x + 3x^2 + 4x^3 + 5x^4)}{(1 + x + x^2 + x^3 + x^4 + x^5)} dx .$$

b) (5 points) A “Trig Trick-or-Treat” problem:

$$\int (1 - x^2)^{-3/2} + (1 - x^2)^{-1/2} + (1 - x^2)^{1/2} dx .$$

**Solution:**

a) Substitute  $u = 1 + x + x^2 + x^3 + x^4 + x^5$  so that we get  $\int du/u = \log(u) + c = \log(1 + x + x^2 + x^3 + x^4 + x^5) + C$ .

b) Use trig substitution  $x = \sin(u)$  in all cases. We get

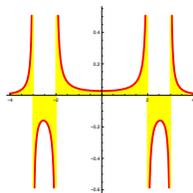
$$\int \frac{1}{\cos^2(u)} + 1 + \cos^2(u) du = \tan(u) + u + (1 + \sin(2u)/2)/2 + C$$

which is  $\tan(\arcsin(x)) + \arcsin(x) + (1 + \sin(2 \arcsin(x)))/2 + C$ .

Problem 10) Partial fractions (10 points)

Integrate

$$\int_{-1}^1 \frac{1}{(x + 3)(x + 2)(x - 2)(x - 3)} dx .$$



The graph of the function is shown to the right.

Lets call it the **friendship graph**.

**Solution:**

Use partial fraction with the Hopital method of course: Write

$$\frac{1}{(x + 3)(x + 2)(x - 2)(x - 3)} = \frac{A}{x + 3} + \frac{B}{x + 2} + \frac{C}{x - 2} + \frac{D}{x - 3}$$

To get  $A$ , multiply the entire equation with  $x + 3$ , simplify and take the limit  $x \rightarrow -3$ . This gives  $A = \frac{1}{(3+3)(3+2)(3-2)} = -1/30$ . Similarly, we get  $B = \frac{1}{(-2+3)(-2+2)(-2-3)} = 1/20$  and  $C = \frac{1}{(-2+3)(-2-2)(-2-3)} = -1/20$  and  $D = \frac{1}{(-3+2)(-3-2)(-3-3)} = 1/30$ . Now we can write the integral as

$$\begin{aligned} & \frac{-1}{30} \log |x + 3| + \frac{1}{20} \log |x + 2| - \frac{1}{20} \log |x - 2| + \frac{1}{30} \log |x - 3| \Big|_{-1}^1 \\ &= \frac{-1}{30} (\log(4) - \log(2)) + \frac{1}{20} \log(3) - \frac{1}{20} \log(3) + \frac{1}{30} (\log(2) - \log(4)) . \end{aligned}$$

Problem 11) Chain rule. (10 points)

a) Find the derivative of

$$f(x) = (\sin(7x + x \cos(x)) - 3x).$$

in general.

b) Now evaluate at  $x = 0$ .

**Solution:**

a)  $\cos(7x + x \cos(x))(-x \sin(x) + \cos(x) + 7) - 3$

b) 5.

Problem 12) Various integration problems (10 points)

a) (2 points)  $\int_0^{2\pi} 2 \cos^2(x) - \sin(x) dx$

b) (2 points)  $\int x^2 e^{3x} dx$

c) (2 points)  $\int_1^{\infty} \frac{1}{(x+2)^2} dx$

d) (2 points)  $\int \sqrt{x} \log(x) dx$

e) (2 points)  $\int_1^e \log(x)^2 dx$

**Solution:**

a) Double angle formula:  $2\pi$

b) Parts (twice)  $e^{3x}(2 - 6x + 9x^2)/27$

c) Substitute  $u = x + 2$  to get  $1/3$

d) Parts differentiating the log  $(3 \log(x) - 2)2x^{3/2}/9$

e) parts writing it as  $\log(x)^2 \cdot 1$  or as  $\log(x) \cdot \log(x)$  (both worked when differentiating the log). The result is  $e - 2$

Problem 13) Applications (10 points)

a) (2 points) [**Agnesi density**]

The CDF of the PDF  $f(x) = \pi^{-1}/(1+x^2)$  is

b) (2 points) [**Piano man**]

The upper hull of  $f(x) = x^2 \sin(1000x)$  is the function

c) (2 points) [**Rower's wisdom**]

If  $f$  is power,  $F$  is work and  $g = F/x$  then  $f = g$  if and only if  $g'(x) =$

d) (2 points) [**Catastrophes**]

For  $f(x) = c(x-1)^2$  there is a catastrophe at  $c =$

e) (2 points) [**Randomness**]

We can use chance to compute integrals. It is called the  method.

**Solution:**

a) Integrate from  $-\infty$  to  $x$  to get  $\arctan(x)/\pi + 1/2$ .

b) The function in front is  $x^2$ . It gives the amplitude.

c) This is the Strawberry theorem applied to an other situation  $g' = 0$ .

d) See where the critical point  $x = 1$  changes nature:  $c = 0$ .

e) This was just a knowledge question: Monte Carlo.