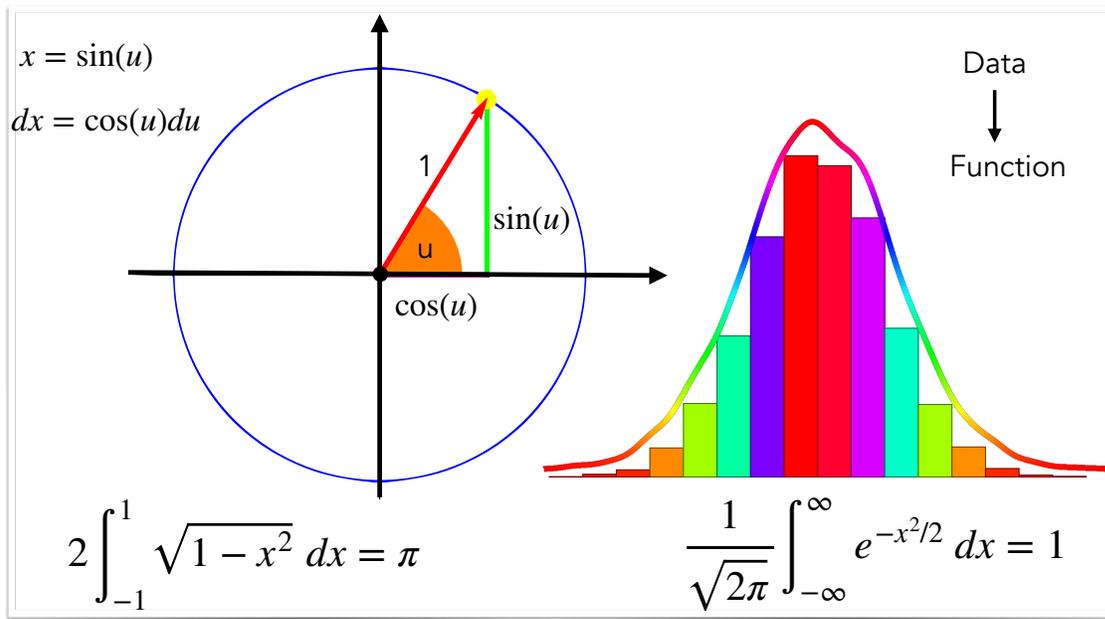


INTRODUCTION TO CALCULUS

OLIVER KNILL



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Introduction

0.1. This course is an introduction to calculus, taught during 2011-2014 and 2020 at Harvard college. No previous calculus exposure is expected, but basic pre-calculus skills like geometry and algebra are expected. Even if you have seen some calculus before, we expect that this course can lead to a deeper, more **conceptual understanding** of the subject and also appreciate its **applications**. Concepts and applications are very important. One should not underestimate however the even more important role of **knowledge** and **skills**. Fortunately, these two are easier to learn and teach than **insight** which requires exponentially more time and also lots of **experience**. Do not worry about it too much at first, as such worries produce learning block. Just enjoy the learning and the doing.

0.2. Our world is **multi-dimensional**. Still, the **one-dimensional** point of view, where we look at the development of a quantity over time, is extremely important. If you study the motion of the universe for example, you are interested in its expansion rate, which is a function of time and so subject to single variable calculus. If you study the spreading of a virus, you are interested in the number of infected as a function of time. If you study climate or weather, you can be interested in the average global temperature over time, like decades or centuries. These are **functions of one variable**, despite the fact that the underlying mechanisms are complex partial differential equations. If you are interested in finance, you can be interested in the stock prize of a single company over time. This is function of variable again.

0.3. While the ideas of calculus already trace back to the time of ancient Greece, the subject has exploded into a **powerful tool** during last few centuries. It is now a pivotal **theoretical foundation** for other mathematical areas and scientific fields. It is no exaggeration to say that calculus is one of most amazing scientific and cultural achievements of humanity.

0.4. Calculus consists of differential and integral calculus. Differential calculus studies “change”, integral calculus deals with “accumulation”. The fundamental theorem of calculus links the two. The subject is very applicable to problems from other scientific disciplines. Calculus is not only important because of its content and applications like life sciences (example: tomography), data science (example: compute correlations), internet (example: networks), artificial intelligence (example: machine learning), geography (example: data visualization), movie and game industry (computer graphics), the ideas of calculus also enter in disguised form, in statistics, economics, computer science, in art or music theory. Do not forget that we primarily want to learn the nuts and bolts and **down-to-earth techniques**.

0.5. The use of computers and computer algebra systems or online tools to experiment with the mathematical structures is encouraged. The use of laptops or tablets in class to take notes is of course perfect. No computer, phone or tablet of any type will be permitted however during exams except possibly as a writing tool, when tests are taken remotely. If you get computer assistance for homework, acknowledge it in the homework. We do recommend that you work out most of the work on paper (or electronic paper). The material sticks better when you write mathematical formulas and procedures by hand. It enhances long term retention and prepares you for exams.

0.6. We do not follow any book or previous course. There are many good books. An example is “Single Variable Calculus: Concepts and Contexts”, by James Stewart. Our homework problems are of similar difficulty than in such textbooks; some of them are trickier and intended to trigger discussion with other students or teaching staff. I recommend to attack each homework problem first on your own. This helps you to develop independent thinking and **problem solving skills**. and also prepares you for the exams. But even if you can solve the problems, it is helpful to discuss them with others.

Oliver Knill, May 10th, 2020.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 1: What is calculus?

LECTURE

1.1. Calculus deals with two things: **taking differences** and **summing things up**. Differences measure **change**, sums quantify how things **accumulate**. The process of taking differences has a limit called the **taking the derivative**. The process of taking sums has a limit called the **taking an integral**. These two operations are related. In this first lecture, we look at functions are evaluated on integers and where no limits are taken. It allows us to illustrate a major benefit of calculus: it gives us the ability to predict the future by analyzing the past.

1.2. Look at the following sequence of numbers

$$1, 7, 17, 31, 49, 71, \dots$$

Can you figure out the next number in this sequence? When solving such riddles, we use already a basic idea of calculus. You might see that the differences

$$6, 10, 14, 18, 22, \dots$$

already show a pattern. Taking differences again gives

$$4, 4, 4, 4, 4, \dots$$

Now, we can go back to the previous sequence and see that that the next term is 26 and going to the original sequence gives $71 + 26 = 97$. Seeing the difference pattern allows us get the future terms in the sequence. This is important.

1.3. Let us rewrite what we just did using the concept of a function. A function f takes an input x and gives something out called $y = f(x)$. The sequence we have just seen can be seen as a function: $f(1) = 1, f(2) = 7, f(3) = 17, f(4) = 31, f(5) = 49, \dots$. Define now a new function Df by $Df(x) = f(x + 1) - f(x)$. It is a rate of change which we also call a “derivative”. Write also $f'(x)$ instead of $f(x)$. We have $f'(1) = 6, f'(2) = 10, f'(3) = 14, \dots$. Now, we can take the derivative again and define $f''(n) = f'(n + 1) - f'(n)$. The function f'' is the function where the derivative has been applied twice. We have seen $f''(1) = 4, f''(2) = 4, f''(3) = 4, \dots$. The second derivative is constant.

1.4. Functions can be visualized graphically in the form of a **graphs** $y = f(x)$. To do so, we draw two perpendicular axes, the x axes and the y axes and mark down every pair $(x, f(x))$ in that Euclidean plane.

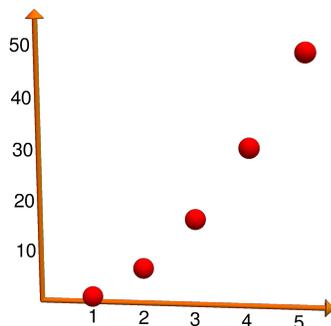


FIGURE 1. When plotting the sequence of numbers in the coordinate plane, we visualize the function as a graph.

1.5. When the first mathematicians were recording numbers they marked them into tally stick. An artifact from tens of thousands of years ago is the **Ishango bone**. We look at this as a constant function

$$1, 1, 1, 1, \dots .$$

Over the next thousands of years, humans figured out to numbers using symbols like

$$1, 2, 3, 4, \dots .$$

We see that $1 = 1, 2 = 1 + 1, 3 = 1 + 1 + 1$ etc. If we look at this counting function $f(x) = x$, it satisfies $f'(x) = 1$, the constant function and $f''(x) = 0$. Which function g has the property that $g' = f$? It is the sum of the terms. For example $f(5) = 0 + 1 + 2 + 3 + 4$ and $f(3) = 0 + 1 + 2 + 3$, then $f(4 + 1) - f(4) = 4$. We see that if we define $g = Sf$ as

$$Sf(x) = f(0) + f(1) + f(2) + \dots + f(x - 1)$$

then $g(x + 1) - g(x) = f(x)$. Can we get a formula for that?

1.6. The new function g satisfies $g(1) = 1, g(2) = 3, g(3) = 6$, etc. These numbers are called **triangular numbers**. From the function g we can get f back by taking difference:

$$Dg(n) = g(n + 1) - g(n) = f(n) .$$

For example $Dg(5) = g(6) - g(5) = 15 - 10 = 5$. And indeed this is $f(5)$. Finding a formula for the sum $Sf(n)$ is not so easy. We have to find the n 'th term in the sequence which starts with

$$1, 3, 6, 10, 15, 21, \dots$$

1.7. Legend tells that when **Karl-Friedrich Gauss** was a 9 year old school kid, his teacher, Mr. Büttner gave him the task to sum up the first 100 positive integers $1 + 2 + \dots + 100$. Gauss did not want to do this tedious work and looked for a better way to do it. He discovered that pairing the numbers up would simplify the summation He would write the sum as $(1 + 100) + (2 + 99) + \dots + (50 + 51)$ so that the answer is $g(x) = x(x - 1)/2 = 5050$. We have now an explicit expression for the sum function. Lets apply the difference function again: $Dg(x) = x(x + 1)/2 - x(x - 1)/2 = x = f(x)$.

1.8. Let us add up the new sequence again and compute $h = Sg$. We get the sequence $0, 1, 4, 10, 20, 35, \dots$ called **tetrahedral numbers**. The reason is that one can use $h(n)$ balls to build a tetrahedron of side length n . For example, we need $h(4) = 20$ golf balls to build a tetrahedron of side length 4. The formula which holds for h is $h(x) = x(x-1)(x-2)/6$. In the worksheet we will check that summing the differences gives the function back.

1.9. The general relation

$$SDf(x) = f(x) - f(0), \quad DSf(x) = f(n)$$

is an arithmetic version of the **fundamental theorem of calculus**. It will lead to the **integral** $\int_0^x f(x) dx$, **derivative** $\frac{d}{dx}f(x)$ and the **fundamental theorem of calculus**

$$\int_0^x \frac{d}{dt}f(t) dt = f(x) - f(0), \quad \frac{d}{dx} \int_0^x f(t) dt = f(x)$$

1.10. It is a fantastic result. The goal of this course is to understand this theorem and apply it. ¹ The above version will lead us. Note that if we define $[n]^0 = 1, [n]^1 = n, [n]^2 = n(n-1)/2, [n]^3 = n(n-1)(n-2)/6$ then $D[n] = [1], D[n]^2 = 2[n], D[n]^3 = 3[n]^2$ and in general

$$\frac{d}{dx}[x]^n = n[x]^{n-1}$$

EXAMPLES

1.11. The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \dots$ satisfies $f(x) = f(x-1) + f(x-2)$. It defines a function on the positive integers. For example, $f(6) = 8$. What is the function $g = Df$? We can assume $f(0) = 0$. **Solution:** We take the difference between successive numbers and get the same sequence again but shifted. We have $Df(x) = f(x-1)$.

1.12. Take the same function f given by the sequence $1, 1, 2, 3, 5, 8, 13, 21, \dots$ but now compute the function $h(n) = Sf(n)$ obtained by summing the first n numbers up. It gives the sequence $1, 2, 4, 7, 12, 20, 33, \dots$. What sequence is that? **Solution:** Because $Df(x) = f(x-1)$ we have $f(x) - f(0) = SDf(x) = Sf(x-1)$ so that $Sf(x) = f(x+1) - f(1)$. Summing the Fibonacci sequence produces the Fibonacci sequence shifted to the left with $f(2) = 1$ is subtracted.

1.13. Find the next term in the sequence $2 \ 6 \ 12 \ 20 \ 30 \ 42 \ 56 \ 72 \ 90 \ 110 \ 132 \ .$ **Solution:** Take differences

2	6	12	20	30	42	56	72	90	110	132	
2	4	6	8	10	12	14	16	18	20	22	.
2	2	2	2	2	2	2	2	2	2	2	

Now we can add an additional number, starting from the bottom and working us up.

2	6	12	20	30	42	56	72	90	110	132	156
2	4	6	8	10	12	14	16	18	20	22	24
2	2	2	2	2	2	2	2	2	2	2	2

¹Many textbooks need hundreds of pages until the fundamental theorem is reached.

1.14. Look at the function $f(n)$ which gives the n 'th prime number. Lets look at the derivatives $D^k f$ but take the absolute value $|D^k(f)|$. In other words, we study $T(f)(n) = |f(n+1) - f(n)|$. We know for example that $f(n) = 2^n$ satisfies $Tf = f$. Lets look at the prime function and apply this differences:

$n=$	1	2	3	4	5	6	7	8	9	...
$f(n) =$	2	3	5	7	11	13	17	23	29	...
$Tf(n) =$	1	2	2	4	2	4	2	4	6	...
$T^2 f(n) =$	1	0	2	2	2	2	2	2	4	...
$T^3 f(n) =$	1	2	0	0	0	0	0	2	0	...

The **Gilbreath conjecture** of 1959 claims that the first entry remains 1 for ever when applying this absolute differentiation process. The problem is still open.

HOMEWORK

This homework is due the next class

Problem 1.1: Predict the future and find the next term in the sequence

2, 10, 30, 68, 130, 222, 350, 520, 738, 1010, 1342, ...

by taking “derivatives” and then “integrating”.

Problem 1.2: Look at the odd numbers $f(n) = 2n + 1$. The sequence starts with 1, 3, 5, 7, 9, 11, 13, ... We want to find a function g which has the property $Dg = f$. To do so, play around and compute $Sf(1) = 1$, $Sf(2) = 1 + 3$, $Sf(3) = 1 + 3 + 5$ etc until you see a pattern. Now guess a formula for $g(n) = Sf(x) = f(0) + f(1) + f(2) + \dots + f(x - 1)$ and verify algebraically that $Dg(x) = g(x + 1) - g(x) = f(x)$ indeed holds.

Problem 1.3: The function $f(x) = 2^x$ is called the **exponential function**. We have for example $f(0) = 1$, $f(1) = 2$, $f(2) = 4$, ... Verify that this function satisfies the equation $Df(x) = f(x)$. The derivative of the exponential function is the exponential function itself.

Problem 1.4: It is believed that the prime function $f(x)$ which gives the x 'th prime has infinitely many values, where $f'(x) = 2$. These are called **prime twins**. Find at least 10 points x , where the “derivative” of the prime function is indeed $f'(x) = 2$.

Problem 1.5: Check in each of the following two cases that $g = Sf$ holds. You can do that by verifying that $Dg = f$.

a) For $f(x) = x + 1$ we have $g(x) = x(x + 1)/2$.

b) For $f(x) = (x - 1)(x - 2)/2$ we have $g(x) = x(x - 1)(x - 2)/6$.

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MATH 1A

Unit 2: Functions

LECTURE

2.1. A **function** is a rule which assigns to a real number a new real number. The function $f(x) = x^3 - 2x$ for example assigns to the number $x = 2$ the value $2^3 - 4 = 4$. A function is given with a **domain** A , the points where f is defined and a **codomain** B a set of numbers which f can reach. Usually, functions are defined everywhere, like the function $f(x) = x^2 - 2x$. If this is the case, we often do not mention the domain or assume that the domain is the place where the function is defined. A function $g(x) = 1/x$ for example can not be evaluated at 0 so that the domain must exclude the point 0. The **inverse** of a function f is a function g such that $g(f(x)) = x$. The function $g(x) = \sqrt{x}$ is the inverse of the function $f(x) = x^2$ on the positive axes $[0, \infty)$.

2.2. Here are a few examples of functions. We will look at them in more detail during the lecture. Very important are polynomials, trigonometric functions, the exponential and logarithmic function. You won't find the h -exponential in any textbook. We will have a bit of fun with them later. If you want take them in the case $h = 1$ and later in the case $h \rightarrow 0$, where it is the usual exponential $\exp(x)$. They are the exponentials and logarithms in "quantum calculus" and will in the limit $h \rightarrow 0$ become the regular exponential and logarithm functions.

constant	$f(x) = 1$	power	$f(x) = 2^x$
identity	$f(x) = x$	exponential	$f(x) = e^x = \exp(x)$
linear	$f(x) = 3x + 1$	logarithm	$f(x) = \log(x) = \ln(x)$
quadratic	$f(x) = x^2$	absolute value	$f(x) = x $
cosine	$f(x) = \cos(x)$	devil comb	$f(x) = \sin(1/x)$
sine	$f(x) = \sin(x)$	bell function	$f(x) = e^{-x^2}$
exponentials	$f(x) = \exp_h(x) = (1 + h)^{x/h}$	Agnesi	$f(x) = \frac{1}{1+x^2}$
logarithms	$f(x) = \exp_h^{-1}$	sinc	$\sin(x)/x$

Here are important functions:

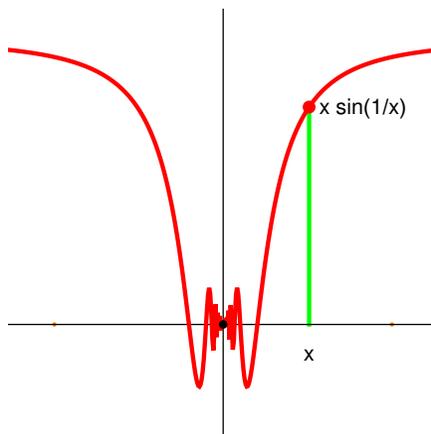
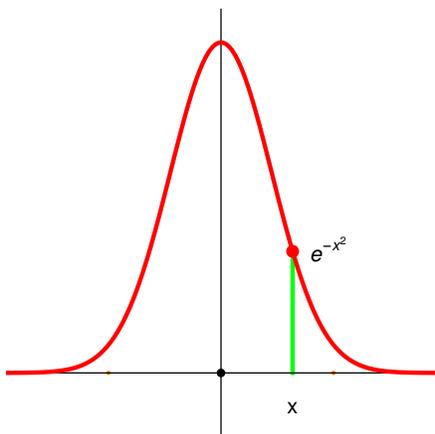
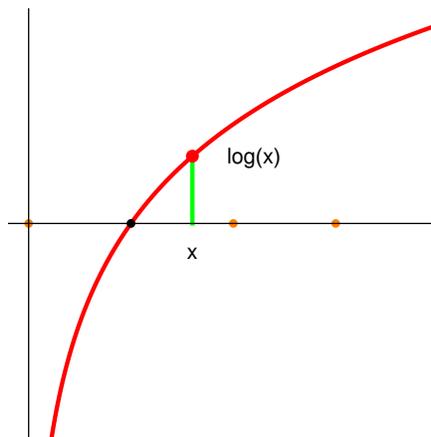
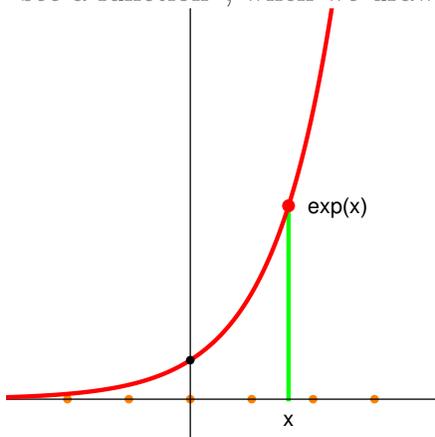
We can build new functions by:

addition	$f(x) + g(x)$
scaling	$2f(x)$
translating	$f(x + 1)$
compose	$f(g(x))$
invert	$f^{-1}(x)$

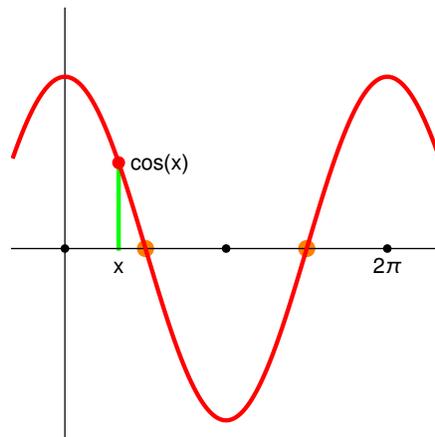
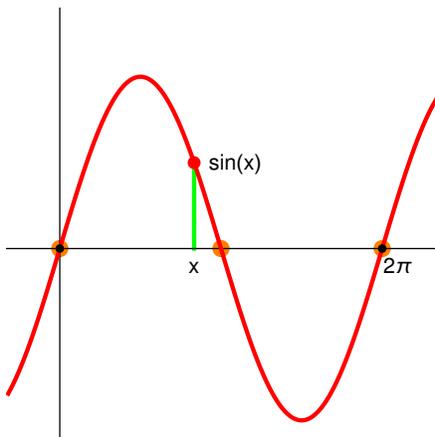
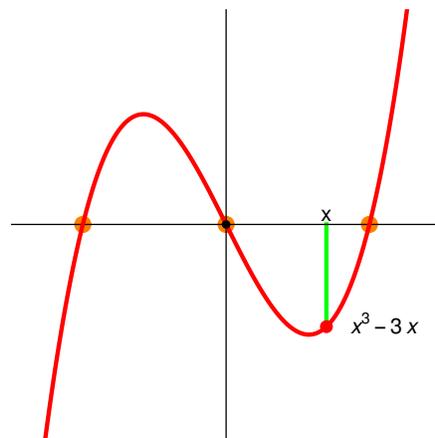
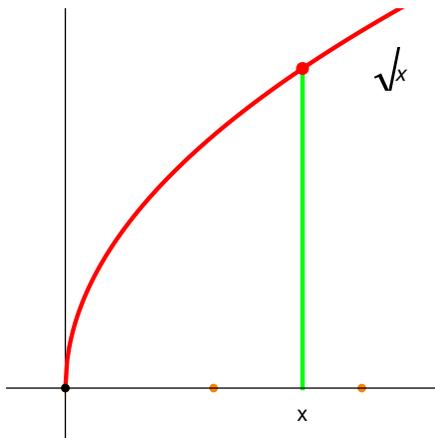
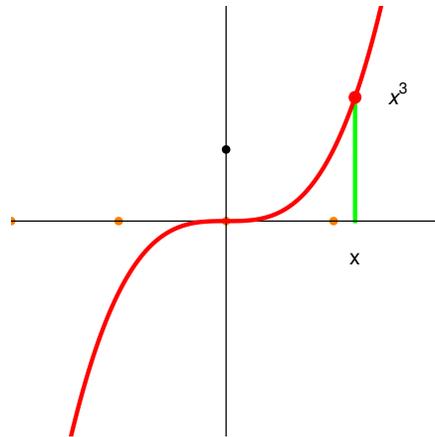
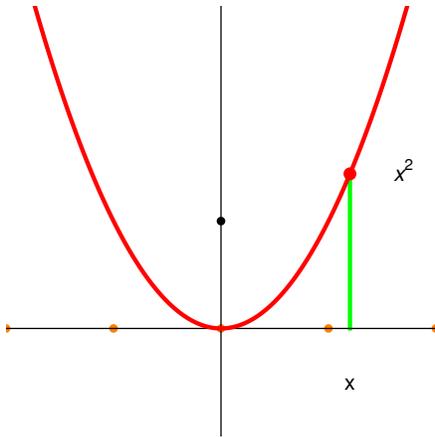
polynomials	$x^2 + 3x + 5$
rational functions	$(x + 1)/(x^4 + 1)$
exponential	e^x
logarithm	$\log(x)$
trig functions	$\sin(x), \tan(x)$
inverse trig functions	$\arcsin(x), \arctan(x)$
roots	$\sqrt{x}, x^{1/3}$

2.3. We will look at these functions **a lot** during this course. The logarithm, exponential and trigonometric functions are especially important. For some functions, we need to restrict the domain, where the function is defined. For the square root function \sqrt{x} or the logarithm $\log(x)$ for example, we have to assume that the number x on which we evaluate the function is positive. We write that the domain is $(0, \infty) = \mathbf{R}^+$. For the function $f(x) = 1/x$, we have to assume that x is different from zero. Keep these three examples in mind.

2.4. The **graph** of a function is the set of points $\{(x, y) = (x, f(x))\}$ in the plane, where x runs over the domain A of f . Graphs allow us to **visualize** functions. We can “see a function”, when we draw the graph.



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2.5. Definition: A function $f : A \rightarrow B$ is **invertible** if there is an other function g such that $g(f(x)) = x$ for all x in A and $f(g(y)) = y$ for all $y \in B$. The function g is the **inverse** of f . Example: $g(x) = \sqrt{x}$ is the inverse of $f(x) = x^2$ as a function from $A = [0, \infty)$ to $B = [0, \infty)$. You can check with the **horizontal line test** whether an inverse exists: draw the box with base A and side B , then every horizontal line should intersect the graph exactly once.

HOMEWORK

Here is the homework for this section.

Problem 2.1: Draw the function $f(x) = x^3 \sin(4x)$ on the interval $[-5, 5]$. Its graph goes through the origin $(0, 0)$. You can use technology.

- A function is called **odd** if $f(-x) = -f(x)$. Is f odd?
- A function is called **even** if $f(-x) = f(x)$. Is f even?
- What happens in general if a function f is both even and odd?

Problem 2.2: Determine from the following functions whether they are invertible. and write down the inverse if they are

- $f(x) = x^7 - 1$ from $A = \mathbb{R}$ to $B = \mathbb{R}$
- $f(x) = \cos(x^3)$ from $A = [0, \pi/2]$ to $B = [0, 1]$
- $f(x) = \sin(x)$ from $A = [0, \pi]$ to $B = [0, 1]$
- $f(x) = \tan(x)$ from $A = (-\pi/2, \pi/2)$ to $B = \mathbb{R}$.
- $f(x) = 1/(1 + x^2)$ from $A = [0, \infty)$ to $B = (0, 1]$.

Problem 2.3: a) Draw the graphs of $\exp_1(x) = 2^x$, $\exp_{1/10}(x) = (1 + 1/10)^{10x}$ and $\exp(x)$.

b) Draw the graphs the inverse of these functions.

You are welcome to use technology for a). For b), just "flip the graph" at the line $x = y$.

Problem 2.4: Try to plot the function $\exp(\exp(\exp(x)))$ on $[0, 1]$. This is a fine function but computer programs do not plot always graph. Describe what you see when the machine plots the function.

Problem 2.5: A function $f(x)$ has a **root** at $x = a$ if $f(a) = 0$. Find at least one root for each of the following functions.

- | | |
|--------------------------|---------------------------------|
| a) $f(x) = \cos(x)$ | d) $f(x) = \log(x) = \ln(x)$ |
| b) $f(x) = 4 \exp(-x^4)$ | e) $f(x) = \sin(x) - 1$ |
| c) $f(x) = x^5 - x^3$ | f) $f(x) = \csc(x) = 1/\sin(x)$ |

(*) Here is how you to plot a function:

[http://www.wolframalpha.com/input/?i=Plot+sin\(x\)](http://www.wolframalpha.com/input/?i=Plot+sin(x))

INTRODUCTION TO CALCULUS

MATH 1A

Unit 3: Limits

LECTURE

3.1. The function $1/x$ is not defined everywhere. It blows up at $x = 0$, because we divide by zero. Sometimes however, functions can be **healed**. A silly example is $f(x) = x^2/x$ which is initially not defined at $x = 0$ because we divide by x but can be “saved” by noticing that $f(x) = x$ for all x different from 0. Functions often can be continued to “forbidden” places if we write the function differently. This can involve dividing out a common factor. Let's look at examples:

3.2. Example. The function $f(x) = (x^3 - 1)/(x - 1)$ is at first not defined at $x = 1$. But for x close to 1, nothing really bad happens. We can evaluate the function at points closer and closer to 1 and get closer and closer to 3. We say $\lim_{x \rightarrow 1} f(x) = 3$. Indeed, you might have noticed that $f(x) = x^2 + x + 1$ by factoring out $(x - 1)$. While initially not defined at $x = 1$, there is a natural value $b = 3$ we can assign for $f(3)$ so that the graph continues nicely through that point.

3.3. Definition. We write $x \rightarrow a$ to indicate that x approaches a . This approach can be from either side. A function $f(x)$ has a **limit** at a point a if there exists a unique b such that $f(x) \rightarrow b$ for $x \rightarrow a$. We write $\lim_{x \rightarrow a} f(x) = b$ if the limit exists and is the same from either side. In whatever way we approach it, we must get the same b .¹

3.4. Example. The function $f(x) = \sin(x)/x$ is called $\text{sinc}(x)$. It is not defined at $x = 0$ at first. It appears naturally in geometry as a quotient between the length of a side of a right angle triangle and an arc length of a sector which contains it. Keep this function in mind. We will look at it later and prove that the limit of $f(x)$ exists for $x \rightarrow 0$. It is so important that it is sometimes called the

Fundamental theorem of trigonometry. $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$.

3.5. Example. The function $f(x) = x/|x|$ is 1 if $x > 0$ and -1 if $x < 0$. It is not defined at $x = 0$ and there is no way to assign a value b at $x = 0$ so that $\lim_{x \rightarrow 0} f(x) = b$. One can define $f(0) = 0$ we can call the function the **sign function**. It is defined everywhere but not continuous at 0.

¹Technical: for all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - b| < \epsilon$.

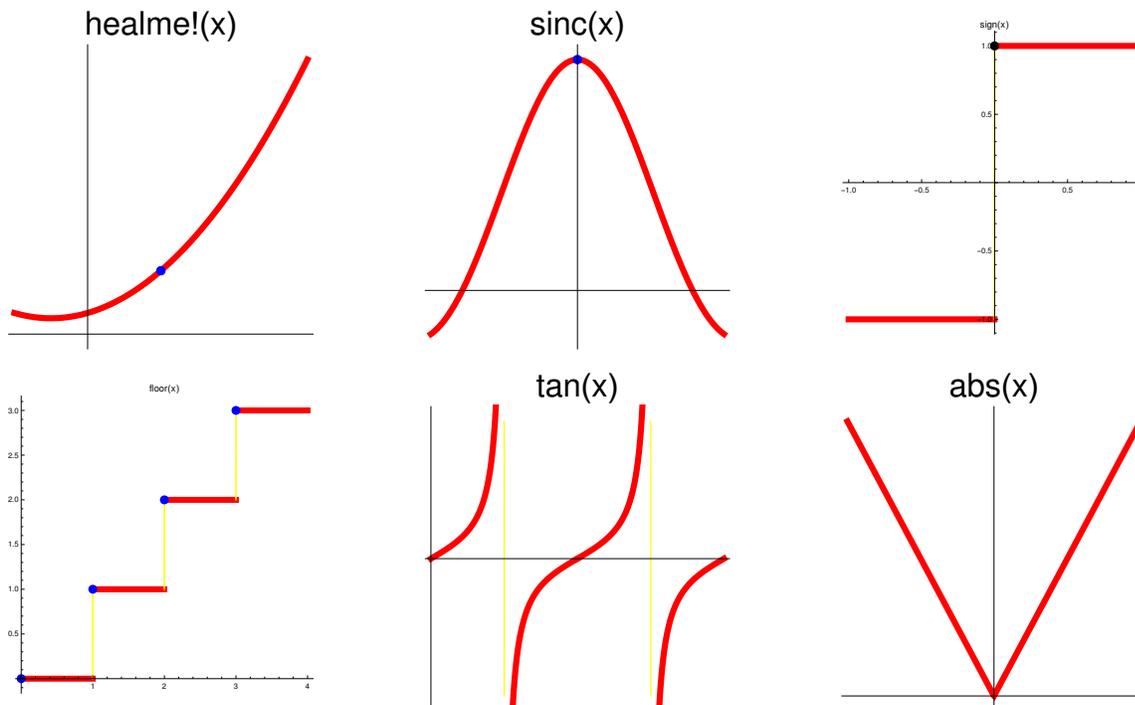


Figure: The graphs $f(x) = (x^3 - 1)/(x - 1)$, the sinc function $\text{sinc}(x) = \sin(x)/x$, the sign function $\text{sign}(x) = x/|x|$, the floor function $\text{floor}(x)$ giving the largest integer smaller or equal to x , the tan function and the **absolute value function** $\text{abs}(x) = |x|$.

3.6. Example. The quadratic function $f(x) = \cos(x^2)/(x^4 + 1)$ has the property that $f(x)$ approaches 1 if x approaches 0. To evaluate functions at 0, there was no need to take a limit because $x^4 + 1$ is never zero. The function is everywhere defined. Actually, most functions are nice in the sense that we do not have to worry about limits at most points. In the overwhelming cases of real applications we only have to worry about limits when the function involves division by 0. For example $f(x) = (x^4 + x^2 + 1)/x$ needs to be investigated more carefully at $x = 0$. You see for example that for $x = 1/1000$, the function is slightly larger than 1000. We can simplify it to $x^3 + x + 1/x$ for $x \neq 0$. There is no limit $\lim_{x \rightarrow 0} f(x)$ because $1/x$ has no limit.

3.7. Example. Also, for \sin and \cos , the limit $\lim_{x \rightarrow a} f(x) = f(a)$ is defined. This extends to **trigonometric polynomials** like $\sin(3x) + \cos(5x)$. The function $\tan(x)$ however has no limit at $x = \pi/2$. No finite value b can be found so that $\tan(\pi/2+h) \rightarrow b$ for $h \rightarrow 0$. This is due to the fact that $\cos(x)$ is zero at $\pi/2$.

3.8. Example. The **cube root** function $f(x) = x^{1/3}$ is defined for all x and even $x = 0$. For the square root function $f(x) = \sqrt{x}$ we have to be aware that for $x < -0$, it is not defined. The domain of the is function is the positive real axis.

Why do we worry about limits at all? One of the main reasons will is that we will soon define the derivative and integral using limits. An other reason is that one can use limits to define numbers like $\pi = 3.1415926 \dots$. In the next lecture, we also look at the important concept of continuity which refers to limits.

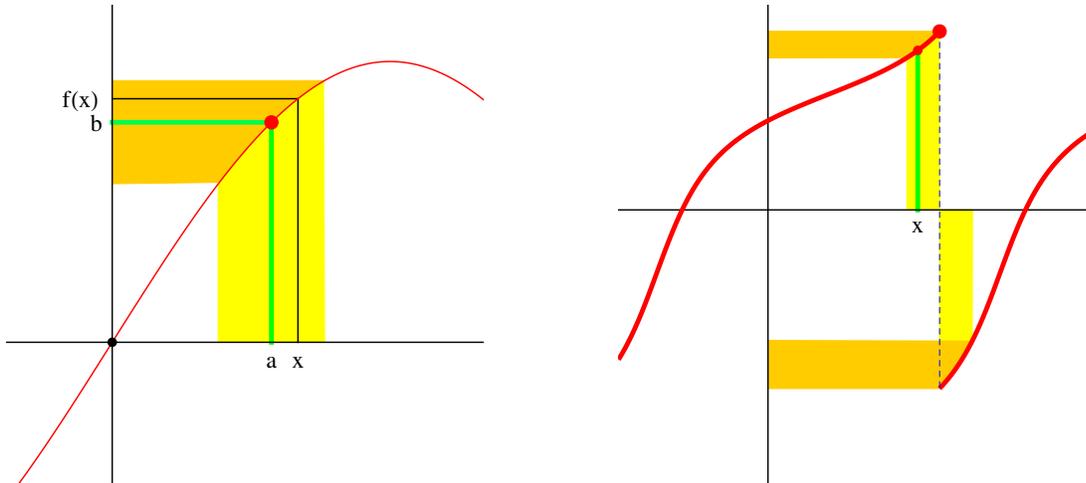


Figure: To the left we see a case, where the limit exists at $x = a$. If x approaches a then $f(x)$ approaches b . To the right we see the function $f(x) = \arctan(\tan(x) + 1)$, where \arctan is the inverse of \tan . The limit does not exist for $a = \pi/2$. If we approach a from the right, we get the limit $-\pi/2$. From the left, we get the limit $f(\pi/2) = \pi/2$. Note that f is not defined at $x = \pi/2$ because $\tan(x)$ becomes infinite there.

Example: Determine from the following functions whether the limits $\lim_{x \rightarrow 0} f(x)$ exist. If it does, find it.

- | | |
|-------------------------------|---------------------------------|
| a) $f(x) = \cos(x)/\cos(2x)$ | b) $f(x) = \tan(x)/x$ |
| c) $f(x) = (x^2 - x)/(x - 1)$ | d) $f(x) = (x^4 - 1)/(x^2 - 1)$ |
| e) $f(x) = (x + 1)/(x - 1)$ | f) $f(x) = x/\sin(x)$ |
| g) $f(x) = 5x/\sin(6x)$ | h) $f(x) = \sin(x)/x^2$ |
| i) $f(x) = \sin(x)/\sin(2x)$ | j) $f(x) = \exp(x)/x$ |

3.9. The following properties hold for limits:

$$\begin{aligned} \lim_{x \rightarrow a} f(x) = b \text{ and } \lim_{x \rightarrow a} g(x) = c &\text{ implies } \lim_{x \rightarrow a} f(x) + g(x) = b + c. \\ \lim_{x \rightarrow a} f(x) = b \text{ and } \lim_{x \rightarrow a} g(x) = c &\text{ implies } \lim_{x \rightarrow a} f(x) \cdot g(x) = b \cdot c. \\ \lim_{x \rightarrow a} f(x) = b \text{ and } \lim_{x \rightarrow a} g(x) = c \neq 0 &\text{ implies } \lim_{x \rightarrow a} f(x)/g(x) = b/c. \end{aligned}$$

3.10. This implies we can sum up and multiply or divide functions which have limits: Examples: Polynomials like $x^5 - 2x + 6$ or trig polynomials like $\sin(3x) + \cos(5x)$ have limits everywhere. Rational functions like $(x^2 - 1)/(x^2 + 1)$ have limits everywhere if the denominator has no roots. Functions like $\cos^2(x) \tan(x)/\sin(x)$ can be healed by simplification. Prototype functions like $\sin(x)/x$ have limits everywhere.

HOMework

Problem 3.1: Find the limits $x \rightarrow 0$. You can use what we have established about $\text{sinc}(x)$.

- | | |
|---------------------------------|--------------------------------|
| a) $f(x) = (x^6 - 1)/(x - 1)$, | b) $f(x) = \sin(23x)/x$ |
| c) $f(x) = \sin^2(9x)/x^2$, | d) $f(x) = \sin(11x)/\sin(7x)$ |

Problem 3.2: a) Graph of the function

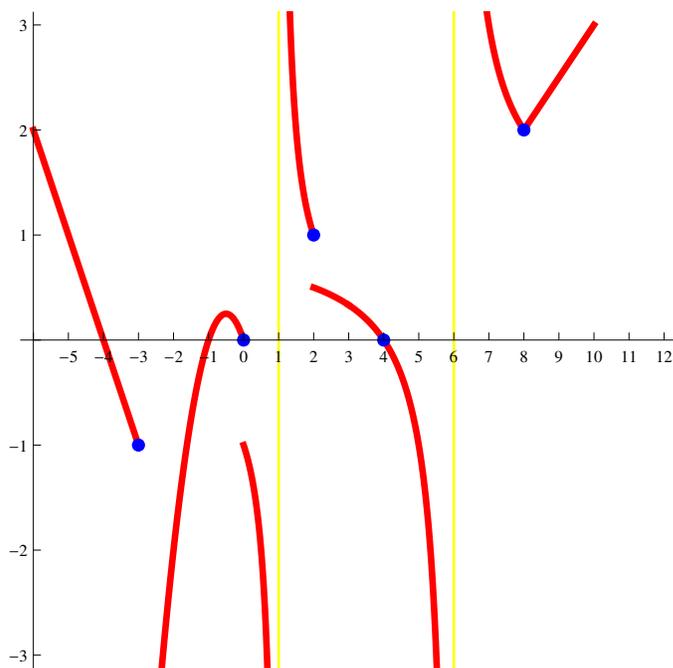
$$f(x) = \frac{(1 - \cos(x))}{x^2}.$$

b) Where is the function f defined? Can you find the limit at the places, where it is not defined? Hint: **remember double angle formulas**

c) Verify that $f(x) = \exp_h(x) = (1 + h)^{x/h}$ satisfies $[f(x + h) - f(x)]/h = f(x)$.

Remark. We define $e^x = \exp(x) = \lim_{h \rightarrow 0} \exp_h(x)$.

Problem 3.3: Find all points a at which the function given in the picture has no limits.



Problem 3.4: Find the limits for $x \rightarrow 0$:

a) $f(x) = (x^2 - 2x + 1)/(x - 1),$

b) $f(x) = \frac{\sin(x) 2^x}{x 5^x}.$

c) $f(x) = 2^{(2^{2^x})},$

d) $f(x) = \frac{\sin(\sin(x))}{\sin(x)}.$

Problem 3.5: We explore in this problem the limit of the function $f(x) = x^x$ if $x \rightarrow 0$. Write a short research paragraph about it. It should involve some experiments and cases. Can we find a limit in general? Take a calculator or use Wolfram α and experiment. What do you see when $x \rightarrow 0$? Can you find an explanation for your experiments? Discuss!

INTRODUCTION TO CALCULUS

MATH 1A

Unit 4: Continuity

LECTURE

4.1. Continuity is one of the most important concepts in mathematics:

Definition: A function f is **continuous** at a point x_0 if a value $f(x_0)$ can be found such that $f(x) \rightarrow f(x_0)$ for $x \rightarrow x_0$. A function f is **continuous on** $[a, b]$ if it is continuous for every point x in the interval $[a, b]$.

4.2. In the interior (a, b) , the limit needs to exist both from the right and from the left. At the boundary a of the interval, only the right limit needs to exist and at the point b , only the left limit. Intuitively, a function is continuous if you can **draw the graph of the function without lifting the pencil**. Continuity means that small changes in x results in small changes of $f(x)$.

4.3. **Example.** Any polynomial like x^3 or trig functions like $\cos(x)$, $\sin(x)$, $\exp(x)$ are continuous everywhere. Also the sum and product of such functions is continuous. For example, $x^5 + \sin(x^3 + x) - x \cos(x^7 + x^2)$ is continuous everywhere. We can also compose functions like $\exp(\sin(x))$ and still get a continuous function.

4.4. The function $f(x) = 1/x$ is continuous everywhere except at $x = 0$. It is a prototype of a function which is not continuous everywhere. The discontinuity at $x = 0$ is also called a **pole**. One can draw a **vertical asymptote**. The **division by zero** kills continuity. Remember however that this can be salvaged in some cases like $f(x) = \sin(x)/x$ which is continuous everywhere. The function can be healed at 0 even so it was at first not defined at $x = 0$.

4.5. The function $f(x) = \log|x|$ is continuous for $x \neq 0$. It is not continuous at 0 because $f(x) \rightarrow -\infty$ for $|x| \rightarrow 0$. It might surprise you that $f(x) = (1 - x^2)/\log|x|$ is continuous everywhere. Yes, it is not defined at first at $x = 0$ as $\log|0| = -\infty$. It is also not defined at $x = 1$ or $x = -1$ at first because $\log(1) = 0$. But in both cases, we can heal. The value $f(0) = 0$ is easier to see, but filling in the value $f(1) = f(-1) = -2$ is less obvious. We will learn this later.

4.6. The co-secant function $\csc(x) = 1/\sin(x)$ is not continuous at $x = 0, x = \pi, x = 2\pi$ and more generally for any multiple of π . It has poles there because $\sin(x)$ is zero there so that we divide by zero at such points. The function $\cot(x) = \cos(x)/\sin(x)$ shares the same discontinuity points as $\csc(x)$.

4.7. The function $f(x) = \sin(\pi/x)$ is continuous everywhere except at $x = 0$. It is a prototype of a function which is not continuous due to **oscillation**. We can approach $x = 0$ in ways that $f(x_n) = 1$ and such that $f(z_n) = -1$. Just chose $x_n = 2/(4k + 1)$ or $z_n = 2/(4k - 1)$.

4.8. The **signum function** $f(x) = \text{sign}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0 \end{cases}$ is not continuous at 0.

It is a prototype of a function with a **jump** discontinuity at 0.

Rules:

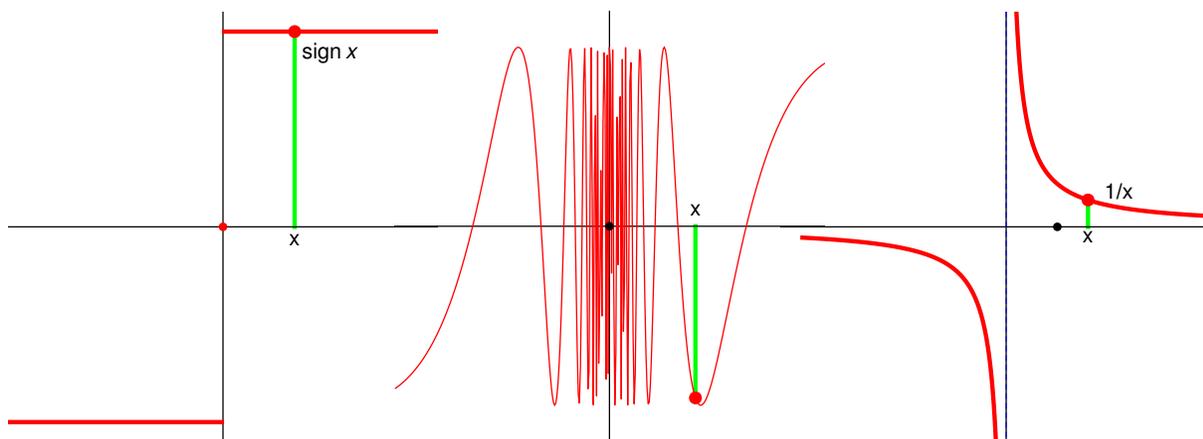
- a) If f and g are continuous, then $f + g$ is continuous.
- b) If f and g are continuous, then $f * g$ is continuous.
- c) If f and g are continuous and if $g > 0$ then f/g is continuous.
- d) If f and g are continuous, then $f \circ g$ is continuous.

Example.

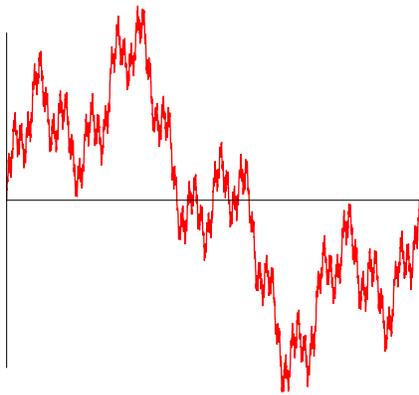
a) $f(x) = \sqrt{x^2 + 1}$ is continuous everywhere on the real line. b) $f(x) = \cos(x) + \sin(x)$ is continuous everywhere. c) $f(x) = \log(|x|)$ is continuous everywhere except at 0. d) $f(x) = \sin(\pi x)/\log|x^4|$ is continuous at $x = 0$. Is it continuous everywhere? Experiment.

Example: The function $f(x) = [\sin(x + h) - \sin(x)]/h$ is continuous for every parameter $h > 0$. We will see soon what happens when h becomes smaller and smaller and that the continuity will not deteriorate. Indeed, we will see that we get closer and closer to the cos function.

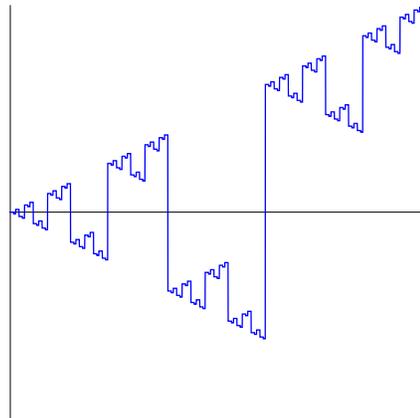
4.9. There are three major reasons, why a function is not continuous at a point: it can **jump**, **oscillate** or **escape** to infinity. Here are the prototype examples. We will look at more during the lecture.



4.10. Why do we like continuity? We will see many reasons during this course but for now lets just say that:



“Continuous functions can be pretty wild, but not too crazy.”



A wild continuous function. This Weierstrass function is believed to be a fractal.

A crazy discontinuous function. It is discontinuous at every point and known to be a fractal.

4.11. Continuity will be useful when finding maxima and minima. A continuous function on an interval $[a, b]$ has a maximum and minimum. We will see in the next hour that if a continuous function is negative at some place and positive at another, there is a point between, where it is zero. Being able to find solutions to equations $f(x) = 0$ is important and much more difficult, if f not continuous.

4.12. Problem Determine for each of the following functions, where discontinuities appear:

- a) $f(x) = \log(|x^2 - 1|)$
- b) $f(x) = \sin(\cos(\pi/x))$
- c) $f(x) = \cot(x) + \tan(x) + x^4$
- d) $f(x) = (x^2 + 2x + 1)/(x + 10 + (x - 1)^2)/(x - 1)$
- e) $f(x) = \frac{x^2 - 4x}{x}$

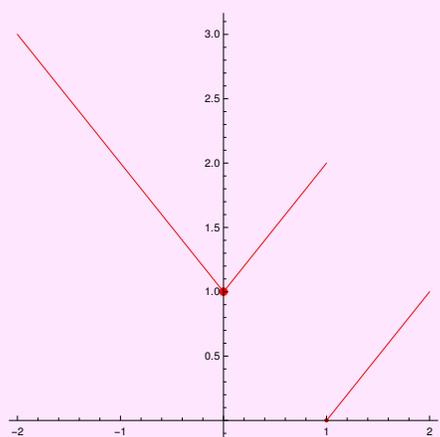
Homework

Problem 4.1: For the following functions, determine the points, where f is not continuous.

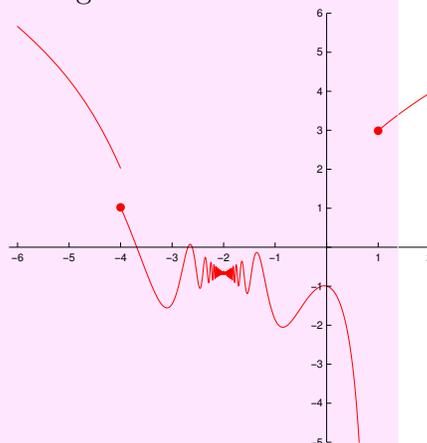
- a) $\text{sinc}(x) + 1/\cos(x)$
- b) $\sin(\tan(x))$
- c) $f(x) = \cot(2 - x)$
- d) $\text{sign}(x)/x$
- e) $\frac{x^2 + 5x + x^4}{x - 3}$

State which kind of discontinuity appears.

Problem 4.2: On which intervals are the following functions continuous?



a)



b)

Problem 4.3: Either do the following three problems a),b),c):

- a) Construct a function which has a jump discontinuity and an escape to infinity.
- b) Find a function which has an oscillatory discontinuity and an escape to infinity.
- c) Find a function which has a jump discontinuity as well as an oscillatory discontinuity.

or shoot down the problem with one strike:

Find a function which has a jump discontinuity, a pole and an oscillatory discontinuity all at the same time.

Problem 4.4: Heal the following functions to make them continuous everywhere

- a) $(x^3 - 8)/(x - 2)$
- b) $(x^5 + x^3)/(x^2 + 3)$
- c) $((\sin(x))^3 - \sin(x))/(\cos(x) \sin(x))$.
- d) $(x^4 + 4x^3 + 6x^2 + 4x + 1)/(x^3 + 3x^2 + 3x + 1)$
- e) $(x^{70} - 1)/(x^{10} - 1)$

Problem 4.5: Are the following function continuous? Break the functions up into simpler functions and analyze each. If you are not sure, experiment by plotting the functions.

- a) $\sin\left(\frac{1}{3+\sin(x)\cos(x)}\right) + |\cos(x)| + \frac{\sin(x)}{x} + x^5 + x^3 + 1 + \frac{7}{\exp(x)}$.
- b) $\frac{2}{\log|x|} + x^7 - \cos(\sin(\cos(x))) - \exp(\log(\exp(x)))$

INTRODUCTION TO CALCULUS

MATH 1A

Unit 5: Intermediate value theorem

LECTURE

5.1. The problem to find solutions to equations can be reduced to finding roots.

Definition: If $f(a) = 0$, then a is called a **root** of f . For $f(x) = \cos(2x)$ for example, there are roots at $x = \pi$ or $x = 3\pi$ or $x = -\pi$.

5.2. Here are a few examples

Example: Find the roots of $f(x) = 4x + 6$. **Answer:** we set $f(x) = 0$ and solve for x . In this case $4x + 6 = 0$ and so $x = -3/2$.

Example: Find the roots of $f(x) = x^2 + 2x + 1$. **Answer:** Because $f(x) = (x + 1)^2$ the function has a root at $x = -1$.

Example: Find the roots of $f(x) = (x - 2)(x + 6)(x + 3)$. **Answer:** Since the polynomial is factored already, it is easy to see the roots $x = 2, x = -6, x = -3$.

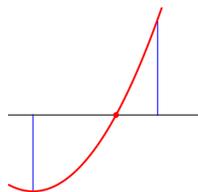
Example: $f(x) = 12 + x - 13x^2 - x^3 + x^4$. Find the roots of f . There is no formula. Just try ($12 = 3 * 4$ is a hint). We see $x = 1, x = -3, x = 4, x = -1$ are the roots.

Example: The function $f(x) = \exp(x)$ does not have any root.

Example: The function $f(x) = \log|x| = \ln|x|$ has roots $x = 1$ and $x = -1$.

Example: $f(x) = 2^x - 16$ has the root $x = 2$.

Intermediate value theorem of Bolzano. If f is continuous on the interval $[a, b]$ and $f(a), f(b)$ have different signs, then there is a root of f in (a, b) .



5.3. The proof is constructive: we can assume $f(a) < 0$ and $f(b) > 0$. The other case is similar. Look at the point $c = (a + b)/2$. If $f(c) < 0$, then look take $[c, b]$ as your new interval, otherwise, take $[a, c]$. We get a new root problem on a smaller interval. Repeat the procedure. After n steps, the search is narrowed to an interval $[u_n, v_n]$ of length $2^{-n}(b - a)$. Continuity assures that $f(u_n) - f(v_n) \rightarrow 0$ and that $f(u_n), f(v_n)$ have different signs. Both point sequences u_n, v_n converge to a root of f .

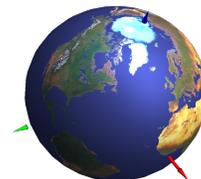
Example: Verify that the function $f(x) = x^{17} - x^3 + x^5 + 5x^7 + \sin(x)$ has a root.

Solution. The function goes to $+\infty$ for $x \rightarrow \infty$ and to $-\infty$ for $x \rightarrow -\infty$. We have for example $f(10000) > 0$ and $f(-1000000) < 0$. Use the theorem.

Example: There is a solution to the equation $x^x = 10$. Solution: for $x = 1$ we have $x^x = 1$ for $x = 10$ we have $x^x = 10^{10} > 10$. Apply the intermediate value theorem.

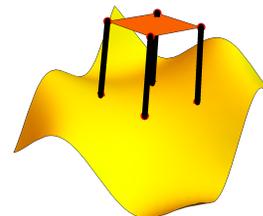
Earth Theorem. There is a point on the earth, where tem-

Example: perature and pressure agrees with the temperature and pressure on the antipode.



Proof. Draw a meridian through the poles and let $f(x)$ be the temperature on that circle. Define $g(x) = f(x) - f(x + \pi)$. If this function is zero on the north pole, we have found our point. If not, $g(x)$ has different signs on the north and south pole. There exists therefore an x , where $g(x) = 0$ and so $f(x) = f(x + \pi)$. For every meridian, we have a latitude value $l(x)$ for which the temperature works. Define now $h(x) = l(x) - l(x + \pi)$. This function is continuous. Start with the meridian 0. If $h(0) = 0$ we have found our point. If not, then $h(0)$ and $h(\pi)$ have different signs. By the intermediate value theorem again, h has a root. There, both temperature and pressure agree with the antipode value.

Example: **Wobbly Table Theorem.** On an arbitrary floor, a square table can be turned so that it does not wobble any more.



Proof. The 4 legs ABCD are located on a square in a plane. Let x be the angle of the line AC with with a coordinate axes if we look from above. Given x , the table can be positioned **uniquely**: the center of ABCD is on the z -axes, the legs ABC are on the floor and AC points in the direction x . Let $f(x)$ denote the height of the fourth leg D from the ground. If we find an angle x such that $f(x) = 0$, we have a position where all four legs are on the ground. Assume $f(0)$ is positive. ($f(0) < 0$ is similar.) Tilt the table around the line AC so that the two legs B,D have the same vertical distance h from the ground. Now translate the table down by h . This does not change the angle x nor the center of the table. The two previously hovering legs BD now touch the ground and the two others AC are below. Now rotate around BD so that the third leg C is on the ground. The rotations and lowering procedures have not changed the location of the center of the table nor the direction. This position is the same as if we had turned the table by $\pi/2$. Therefore $f(\pi/2) < 0$. The intermediate value theorem assures that f has a root between 0 and $\pi/2$.

Definition: Lets call $Df(x) = (f(x+h) - f(x))/h$ the **h -derivative** of f . We will study it more in the next lecture. You have in a homework already verified that $D \exp_h(x) = \exp_h(x)$.

Definition: Lets call a point p , where $Df(x) = 0$ a **h -critical point**. Lets call a point a a **local maximum** if $f(a) \geq f(x)$ in an open interval containing a . Define similarly a **local minimum** as a point where $f(a) \leq f(x)$.

Example: The function $f(x) = x(x-h)(x-2h)$ has the derivative $Df(x) = 3x(x-h)$ as you have verified in the case $h = 1$ in the first lecture of this course in a worksheet. We will write $[x]^3 = x(x-h)(x-2h)$ and $[x]^2 = x(x-h)$. The computation just done tells that $D[x]^3 = 3[x]^2$. Since $[x]^2$ has exactly two roots $0, h$, the function $[x]^3$ has exactly 2 critical points.

Example: More generally for $[x]^{n+1} = x(x-h)(x-2h) \cdots (x-nh)$ we have $D[x]^{n+1} = (n+1)D[x]^n$. Because $[x]^n$ has exactly n roots, the function $[x]^{n+1}$ has exactly n critical points. Keep the formula

$$D[x]^n = n[x]^{n-1}$$

in mind!

Example: You have verified that $\exp_h(x) = (1+h)^{x/h}$ satisfies $D \exp_h(x) = \exp_h(x)$. Because this function has no roots and the derivative is the function itself, the function has no critical points.

$$D \exp(x) = \exp(x)$$

In our discrete setting we have a statement which later will be repeated when using actual critical points and when f has more regularity.

Fermat's maximum theorem If f is continuous and has a h -critical point a , then f has either a local maximum or local minimum inside the open interval $(a, a+h)$.

5.4. Look at the range of the function f restricted to $[a, a+h]$. It is a bounded interval $[c, d]$ because f is continuous. There exists especially a point u for which $f(u) = c$ and a point v for which $f(v) = d$. These points are different if f is not constant on $[a, a+h]$ and $b = f(a) = f(a+h)$ are in that interval. If $b = c$, then we have at least one local maximum v in $[a, a+h]$ if $b = d$ we have at least one local minimum u in $[a, a+h]$ if $c < b < d$, then there is both a local maximum v and a local minimum u in $[a, a+h]$. We will later argue with the intermediate value theorem.

Example: Problem. Verify that a quadratic polynomial has exactly one critical point. **Solution** $f(x) = ax^2 + bx + c$ with $a \neq 0$. Because the x^2 terms cancel, $f(x+h) - f(x)$ is a linear function $(2ah)x + (bh + ah^2)$. There is a root $x = (-b - ah)/(2a)$.

5.5. What we denote by “ h -critical point” will in the limit $h \rightarrow 0$ be called “critical point” later in this course. While the h -critical point notion makes sense for all continuous functions. We will need more regularity to take the limit $h \rightarrow 0$. This limit $h \rightarrow 0$ will be considered in the future and functions for which the limit exists will be called **differentiable**. But we are not yet there.

HOMEWORK

Problem 5.1: Find the roots for $-x^5 - 3x^4 + 42x^3 + 62x^2 - 297x - 315$. You are told that all roots are integers.

Problem 5.2: Use the intermediate value theorem to verify that $f(x) = x^7 - 6x^6 + 8$ has at least two roots on $[-2, 2]$.

Problem 5.3: Motivated by the superbowl show: Beyoncé height is 169 cm. Shakira's height is 157 cm. Beyoncé was born September 4, 1981, Shakira was born February 2nd, 1977. Beyoncé is believed to be worth 400 million and Shakira's net worth is believed to be 300 million.

- Can you argue that there was a moment when Shakira's height was exactly Beyoncé's height?
- Can you argue that there was a moment when Shakira's age was exactly twice the age of Beyoncé?
- Can you argue that there was a moment when Shakira's fortune was exactly half of Beyoncé's fortune?
- Assume you live in New York. Show that if you drive the 190 miles from Boston to New York in 4 hours then there are at least two moments of time when you drive with exactly 40 miles per hours. The trip is not part of a larger trip. Your start is in Boston and end in New York.



Problem 5.4: Argue why there is a solution to

a) $5 - \sin(x) = x$, b) $\exp(7x) = x$, c) $\sin(x) = x^4$.

d) Why does the following argument not work:

The function $f(x) = 1/\cos(x)$ satisfies $f(0) = 1$ and $f(\pi) = -1$. There exists therefore a point x where $f(x) = 0$.

e) Does the function $f(x) = x + \log|\log|x||$ have a root somewhere?

Problem 5.5: a) Find a concrete function f which has three local maxima and two local minima.

b) Let $h = 1/10$. Find a h -critical point for the function $f(x) = |x|$, that is a point for which $[f(x+h) - f(x)]/h = 0$.

c) Verify that for any $h > 0$, the function $f(x) = x^5$ has no h -critical point. There is no x, h where $[f(x+h) - f(x)]/h = 0$ is possible.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 6: Fundamental theorem

LECTURE

6.1. **Calculus** is a theory of **differentiation** and **integration**. We explore here this concept again in a simple setup and practice differentiation and integration **without taking limits**. We fix a positive constant h and take differences and sums. The fundamental theorem of calculus for $h = 1$ generalizes. We can then differentiate and integrate polynomials, exponentials and trigonometric functions. Later, we will do the same with actual derivatives and integrals. But now, we can work with arbitrary continuous functions. The constant h never pops up. Thinking of it as something fixed, like the God-given **Planck constant** $1.6 \cdot 10^{-35}m$. In the standard calculus of Newton and Leibniz the limit $h \rightarrow 0$ is taken.

Definition: Given $f(x)$, define the **difference quotient**

$$Df(x) = \frac{f(x+h)-f(x)}{h}$$

6.2. If f is continuous then Df is a continuous. For shorthand, we call it simply the "derivative". We keep h constant and positive here. As an example, let's take the **constant function** $f(x) = 5$. We get $Df(x) = (f(x+h) - f(x))/h = (5 - 5)/h = 0$ everywhere. You can see that in general, if f is a constant function, then $Df(x) = 0$.

6.3. $f(x) = 3x$. We have $Df(x) = (f(x+h) - f(x))/h = (3(x+h) - 3x)/h$ which is $\boxed{3}$. You see in general that if $f(x) = mx + b$, then $Df(x) = \boxed{m}$.

For constant functions, $Df(x) = 0$. For linear functions, $Df(x)$ is the slope.

6.4. For $f(x) = x^2$ we compute $Df(x) = ((x+h)^2 - x^2)/h = (2hx + h^2)/h = \boxed{2x + h}$.

6.5. Given a function f , define a new function $Sf(x)$ by summing up all values of $f(jh)$, where $0 \leq jh < x$:

Definition: Given $f(x)$ define the **Riemann sum**

$$Sf(x) = h[f(0) + f(h) + f(2h) + \cdots + f((k-1)h)]$$

In short hand, we call Sf also the "integral" or "anti-derivative" of f . It will become the integral in the limit $h \rightarrow 0$ later in the course.

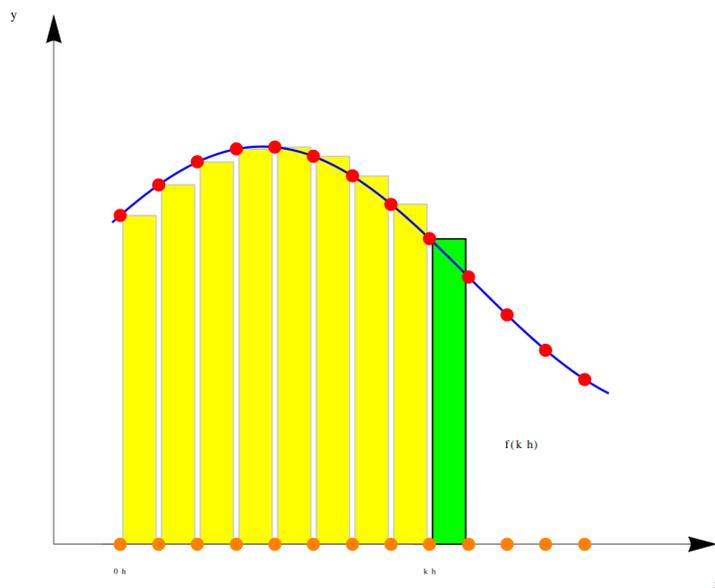
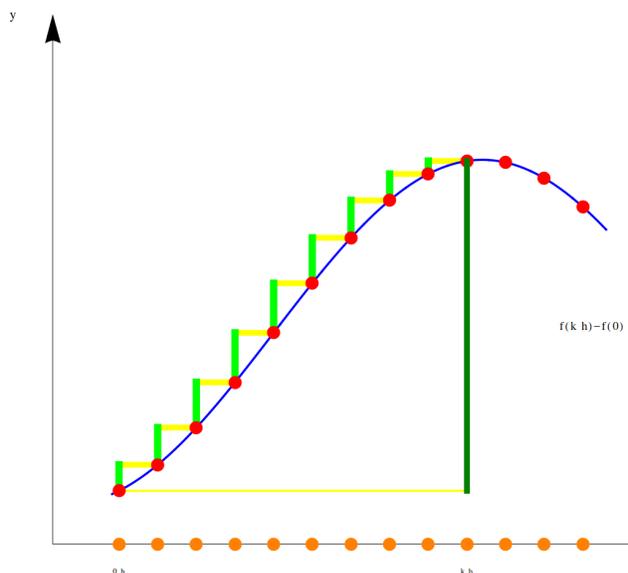
6.6. Compute $Sf(x)$ for $f(x) = 1$. **Solution.** We have $Sf(x) = 0$ for $x \leq h$, and $Sf(x) = h$ for $h \leq x < 2h$ and $Sf(x) = 2h$ for $2h \leq x < 3h$. In general $S1(jh) = j$ and $S1(x) = kh$ where k is the largest integer such that $kh < x$. The function g grows linearly but grows in quantized steps.

The difference $Df(x)$ will become the **derivative** $f'(x)$.

The sum $Sf(x)$ will become the **integral** $\int_0^x f(t) dt$.

Df means **rise over run** and is close to the **slope** of the graph of f .

Sf means **areas of rectangles** and is close to the **area** under the graph of f .



6.7. Here is the **quantum fundamental theorem of calculus**

Theorem: Sum the differences and get

$$SDf(kh) = f(kh) - f(0)$$

Theorem: Difference the sum and get

$$DSf(kh) = f(kh)$$

Example: For $f(x) = [x]_h^m = x(x-h)(x-2h)\dots(x-mh+h)$ we have

$$f(x+h) - f(x) = (x(x-h)(x-2h)\dots(x-kh+2h))((x+h) - (x-mh+h)) = [x]^{m-1}hm$$

and so $D[x]_h^m = m[x]_h^{(m-1)}$. We have obtained the important formula $D[x]^m = m[x]^{m-1}$

6.8. We can establish from this differentiation formulas for **polynomials**. We will leave away the square brackets later to make it look like the calculus we will do later on. In the homework, we already use the usual notation.

6.9. If $f(x) = [x] + [x]^3 + 3[x]^5$ then $Df(x) = 1 + 3[x]^2 + 15[x]^4$. The fundamental theorem allows us to integrate and get $Sf(x) = [x]^2/2 + [x]^4/4 + 3[x]^6/6$.

Definition: Define $\exp_h(x) = (1+h)^{x/h}$. It is equal to 2^x for $h = 1$ and morphs into the function e^x when h goes to zero.

Problem 6.3: Find the derivatives $Df(x)$ of the following functions
 a) $f(x) = \exp(9 \cdot x + 3) + x^6$ b) $f(x) = 8 \exp(-3 \cdot x) + 9x^6$
 c) $f(x) = \exp(6 \cdot x) + \log/(1 + x)$ d) $\log(1 - x^2)$

Problem 6.4: a) Assume $h = 1/100$. Use Wolfram alpha to plot $\cos_h(x)$ and $\sin_h(x)$ on the interval $[-2\pi, 2\pi]$. **Hint.** This means you have to plot the real and imaginary part of $(1 + i/100)^{100x}$. If you enter the expression into Wolfram alpha, it will plot the real and imaginary part.
 b) Do the same for $h = 1/1000$. What has changed?

Problem 6.5: a) Write down again on your own that if $f(x) = (1+h)^{x/h}$, then $Df(x) = f(x)$. (We have done this already twice. Do it again!).
 b) Write down again on your own that if $f(x) = x(x-h)(x-2h)(x-3h)$, then $Df(x) = 4x(x-h)(x-2h)(x-3h)$ meaning $D[x]^4 = 4[x]^3$.

6.14. In this unit we used a self-contained calculus which does not refer to any limit. It is **quantum calculus** and like the calculus we will see when $h \rightarrow 0$, where the formulas are **all the same**. So, why do we not stop here, call it a day, and have an extended spring break until May 2020? Because we want also to have fun in the limit $h \rightarrow 0$!

Fundamental theorem of Calculus: $DSf(x) = f(x)$ and $SDf(x) = f(x) - f(0)$.

Differentiation rules

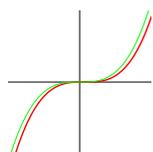
- $Dx^n = nx^{n-1}$
- $De^{a \cdot x} = ae^{a \cdot x}$
- $D \cos(a \cdot x) = -a \sin(a \cdot x)$
- $D \sin(a \cdot x) = a \cos(a \cdot x)$
- $D \log(x + a) = 1/(x + a)$

Integration rules (for $x = kh$)

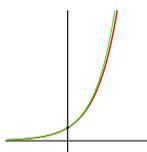
- $Sx^n = x^{n+1}/(n + 1)$
- $Se^{a \cdot x} = (e^{a \cdot x} - 1)/a$
- $S \cos(a \cdot x) = \sin(a \cdot x)/a$
- $S \sin(a \cdot x) = -\cos(a \cdot x)/a$
- $S \frac{1}{x+a} = \log(x + a) - \log(a)$

Fermat's extreme value theorem: If $Df(x) = 0$ and f is continuous, then f has a local maximum or minimum in the open interval $(x, x + h)$.

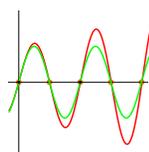
Pictures



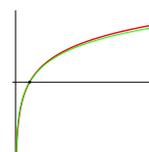
$[x]_h^3$ for $h = 0.1$



$\exp_h(x)$ for $h = 0.1$



$\sin_h(x)$ for $h = 0.1$



$\log_h(x)$ for $h = 0.1$

INTRODUCTION TO CALCULUS

MATH 1A

Unit 7: Rate of Change

LECTURE

7.1. Given a function f and a constant $h > 0$, we can look at the new function

$$Df(x) = \frac{f(x+h) - f(x)}{h}.$$

It is the **rate of change** of the function with **step size** h . When changing x to $x+h$ and then $f(x)$ changes to $f(x+h)$. The quotient $Df(x)$ is a **slope** and “**rise over run**”. In this lecture, we take the limit $h \rightarrow 0$ and derive the important formulas $\frac{d}{dx}x^n = nx^{n-1}$, $\frac{d}{dx}\exp(x) = \exp(x)$, $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$ which we have seen already before in a discrete setting. But now we take the limit $h \rightarrow 0$:

Definition: If the limit $\frac{d}{dx}f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ exist, we say f is **differentiable** at the point x . The value is called the **derivative** or **instantaneous rate of change** of the function f at x . We denote the limit also with $f'(x)$.

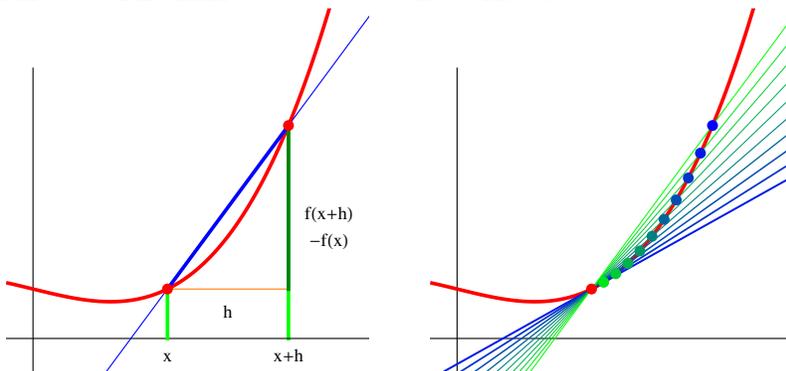
7.2. Example. For $f(x) = 30 - x^2$ we have

$$f(x+h) - f(x) = [30 - (x+h)^2] - [30 - x^2] = -2xh - h^2$$

Dividing this by h gives $-2x - h$. The limit $h \rightarrow 0$ gives $-2x$. We have just seen that for $f(x) = x^2$, we get $f'(x) = -2x$. For $x = 3$, this is -6 .

Example. For $f(x) = |x|$, we have $(f(x+h) - f(x))/h = 1$ if $x > 0$ and $(f(x+h) - f(x))/h = -1$ if x is smaller than $-h$. The limit $h \rightarrow 0$ does not exist at $x = 0$!

The derivative $f'(x)$ has a geometric meaning. It is the slope of the tangent at x . This is an important geometric interpretation. It is useful to think about x as “time” and the derivative as the rate of change of the quantity $f(x)$ in time.



For $f(x) = x^n$, we have $f'(x) = nx^{n-1}$.

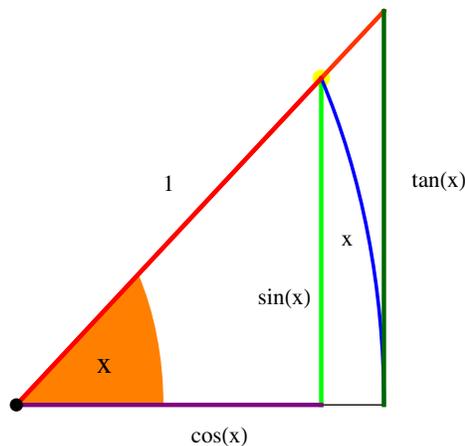
Proof: $f(x+h) - f(x) = (x+h)^n = (x^n + nx^{n-1}h + a_2h^2 + \dots + h^n) - x^n = nx^{n-1}h + a_2h^2 + \dots + h^n$. If we divide by h , we get $nx^{n-1} + h(a_2 + \dots + h^{n-2})$ for which the limit $h \rightarrow 0$ exists: it is nx^{n-1} . This is an important result because many functions can be approximated very well with polynomials.

For $f(x) = \sin(x)$ we have $f'(0) = 1$ because the differential quotient is $[f(0+h) - f(0)]/h = \sin(h)/h = \text{sinc}(h)$. We have already mentioned that the limit is 1 before. Lets look at it again geometrically. For all $0 < x < \pi/2$ we have

$$\sin(x) \leq x \leq \tan(x) .$$

Now divide everything by $\sin(x)$. Because $\tan(x)/\sin(x) = 1/\cos(x) \rightarrow 1$ for $x \rightarrow 0$, the value of $\text{sinc}(x) = \sin(x)/x$ must go to 1 as $x \rightarrow 0$. Renaming the variable x with the variable h , we have verified the **fundamental theorem of trigonometry**

$$\lim_{h \rightarrow 0} \frac{\sin(h)}{h} = 1$$



7.3. For $f(x) = \cos(x)$ we have $f'(x) = 0$. To see this, look at $f(0+h) - f(0) = \cos(h) - 1$. Geometrically, we can use Pythagoras $\sin^2(h) + (1 - \cos(h))^2 \leq h^2$ to see that $2 - 2\cos(h) \leq h^2$ or $(1 - \cos(h)) \leq h^2/2$ so that $(1 - \cos(h))/h \leq h/2$. This goes to 0 for $h \rightarrow 0$. We have just nailed down another important identity

$$\lim_{h \rightarrow 0} \frac{(1 - \cos(h))}{h} = 0 .$$

The interpretation is that the tangent is **horizontal** for the \cos function at $x = 0$. We will call this a critical point later on.

7.4. From the previous two examples, we get

$$\cos(x+h) - \cos(x) = \cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x) = \cos(x)(\cos(h) - 1) - \sin(x)\sin(h)$$

because $(\cos(h) - 1)/h \rightarrow 0$ and $\sin(h)/h \rightarrow 1$, we see that $[\cos(x+h) - \cos(x)]/h \rightarrow -\sin(x)$.

For $f(x) = \cos(ax)$ we have $f'(x) = -a \sin(ax)$.

7.5. Similarly,

$$\sin(x+h) - \sin(x) = \cos(x)\sin(h) + \sin(x)\cos(h) - \sin(x) = \sin(x)(\cos(h) - 1) + \cos(x)\sin(h)$$

because $(\cos(h) - 1)/h \rightarrow 0$ and $\sin(h)/h \rightarrow 1$, we see that $[\sin(x+h) - \sin(x)]/h \rightarrow \cos(x)$.

for $f(x) = \sin(ax)$, we have $f'(x) = a \cos(ax)$.

$$e^x = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{nx}$$

7.6. Like π , the **Euler number** $e = e^1$ is irrational. Here are the first digits: 2.7182818284590452354. If you want to find an approximation, just pick a large n , like $n = 100$ and compute $(1 + 1/n)^n$. For $n = 100$ for example, we see $101^{100}/100^{100}$. We only need 101^{100} and then put a comma after the first digit to get an approximation.

7.7. To see why the limit exists: verify that the fractions $A_n = (1 + 1/n)^n$ increase and $B_n = (1 + 1/n)^{(n+1)}$ decrease. Since $B_n/A_n = (1 + 1/n)$ which goes to 1 for $n \rightarrow \infty$, the limit exists. The same argument shows that $(1 + 1/n)^{xn} = \exp_{1/n}(x)$ increases and $\exp_{1/n}(x)(1 + 1/n)$ decreases. The limiting function $\exp(x) = e^x$ is called the **exponential function**. Remember that if we write $h = 1/n$, then $(1 + 1/n)^{nx} = \exp_h(x)$ considered earlier in the course. We can sandwich the exponential function between $\exp_h(x)$ and $(1 + h)\exp_h(x)$:

$$\exp_h(x) \leq \exp(x) \leq \exp_h(x)(1 + h), \quad x \geq 0 .$$

For $x < 0$, the inequalities are reversed.

7.8. Lets compute the derivative of $f(x) = e^x$ at $x = 0$. **Answer.** We have for $x \leq 1$

$$1 \leq (e^x - 1)/x \leq 1 + x .$$

Therefore $f'(0) = 1$. The exponential function has a graph which has slope 1 at $x = 0$.

7.9. Now, we can get the general case. It follows from $e^{x+h} - e^x = e^x(e^h - 1)$ that the derivative of $\exp(x)$ is $\exp(x)$.

For $f(x) = \exp(ax)$, we have $f'(x) = a \exp(ax)$.

7.10. It follows from the properties of taking limits that $(f(x) + g(x))' = f'(x) + g'(x)$. We also have $(af(x))' = af'(x)$. From this, we can now compute many derivatives

7.11. Find the slope of the tangent of $f(x) = \sin(3x) + 5 \cos(10x) + e^{5x}$ at the point $x = 0$. **Solution:** $f'(x) = 3 \cos(3x) - 50 \sin(10x) + 5e^{5x}$. Now evaluate it at $x = 0$ which is $3 + 0 + 5 = 8$.

Finally, lets mention an example of a function which is not everywhere differentiable.

7.12. The function $f(x) = |x|$ has the properties that $f'(x) = 1$ for $x > 0$ and $f'(x) = -1$ for $x < 0$. The derivative does not exist at $x = 0$ even so the function is continuous there. You see that the slope of the graph jumps discontinuously at the point $x = 0$.

7.13. For a function which is discontinuous at some point, we don't even attempt to differentiate it there. For example, we would not even try to differentiate $\sin(4/x)$ at $x = 0$ nor $f(x) = 1/x^3$ at $x = 0$ nor $\sin(x)/|x|$ at $x = 0$. Remember these bad guys?

To the end, you might have noticed that in the boxes, more general results have appeared, where x is replaced by ax . We will look at this again but in general, the relation $f'(ax) = af'(ax)$ holds ("if you drive twice as fast, you climb twice as fast").

Homework

Problem 7.1: For which of the following functions does the derivative $f'(x)$ exist for every x ?

- | | | |
|-----------------|------------------|----------------------------------|
| a) $ \sin(6x) $ | b) $ 3 \exp(x) $ | c) $4x + \exp(7x) + 3 \sin(45x)$ |
| d) $\sin^6(x)$ | e) $\sin(4/x)$ | f) $\exp(-x) + 1 + \cos(15x) $ |

Problem 7.2: a) A circle of radius r encloses a disc of area $f(r) = \pi r^2$. Find $\frac{d}{dr}f(r)$. Evaluate the rate of change at $r = 1/10$.

b) The ball of radius r has the volume $f(r) = 4\pi r^3/3$. Find $\frac{d}{dr}f(r)$ at $r = 1/10$ and compare it with the surface area of the sphere bounding the ball.

c) A **hypersphere** of radius r has the **hyper volume** $f(r) = \pi^2 r^4/2$. Find $\frac{d}{dr}f(r)$ and evaluate it at $r = 1/10$.

Problem 7.3: Find the derivatives of the following functions at the point $x = 0$.

- a) $f(x) = 7 \exp(4x) + \sin(11x) + x + x^4 + x^6 + x^{10}$.
- b) $f(x) = (x^5 - 1)/(x - 1) + \cos(2x)$. First heal this function.
- c) $f(x) = \frac{1+4x+6x^2+4x^3+x^4}{x^2+2x+1}$. Also here, first heal!

Problem 7.4: In this problem we compute the derivative of \sqrt{x} for $x > 0$. To do so, we have to find the limit

$$\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}.$$

Hint: multiply the top and the bottom with $(\sqrt{x+h} + \sqrt{x})$ then use algebra to simplify.

Problem 7.5: While watching the movie “Aeronauts”, we assume that the height $h(t)$ of a balloon $h(t) = \sqrt{t^2 + 20^2}$. Compute the change of $h(t)$ at $t = 0$ and plot the function $h(t)$.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 8: Derivative Function

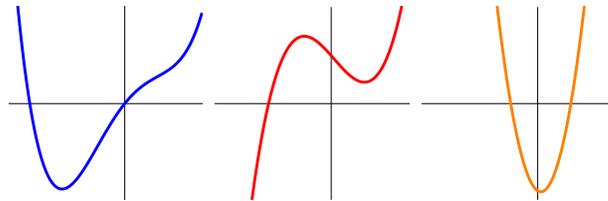
LECTURE

8.1. We have defined the derivative $f'(x) = \frac{d}{dx}f(x)$ as a limit of $(f(x+h) - f(x))/h$ for $h \rightarrow 0$. We have seen that $\frac{d}{dx}x^n = nx^{n-1}$ holds for integer n . We also know already that $\sin'(ax) = a \cos(x)$, $\cos'(ax) = -a \sin(x)$ and $\exp'(ax) = a \exp(x)$. We can already differentiate a lot of functions and evaluate the derivative $f'(x)$ at some point x and so compute the slope of the curve at x .

8.2. Example: Find the derivative $f'(x)$ of $f(x) = \sin(4x) + \cos(5x) - \sqrt{x} + 1/x + x^4$ and evaluate it at $x = 1$. **Solution:** $f'(x) = 4 \cos(4x) - 5 \sin(5x) - 1/(2\sqrt{x}) - 1/x^2 + 4x^3$. Plugging in $x = 1$ gives $-\pi - 1/2 - 1 + 4$.

8.3. One can look at the differentiation process also as a rule which assigns to a function f a new function f' , the **derivative function**. For example, for $f(x) = \sin(x)$, we get $f'(x) = \cos(x)$. In this lecture, we want to understand the new function and its relation with f . What does it mean if $f'(x) > 0$. What does it mean that $f'(x) < 0$. Do the roots of f tell something about f' or do the roots of f' tell something about f ?

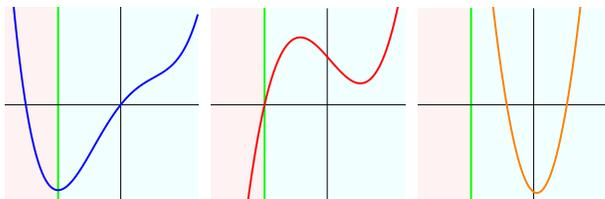
8.4. Here is an example of a function f , its derivative f' and the derivative of the derivative f'' . Can you see the relation?



8.5. To understand the relation, it is good to distinguish intervals, where $f(x)$ is increasing or decreasing. These are the intervals where $f'(x)$ is positive or negative.

Definition: A function is called **monotonically increasing** on an interval $I = (a, b)$ if $f'(x) > 0$ for all $x \in (a, b)$. It is **monotonically decreasing** if $f'(x) < 0$ for all $x \in (a, b)$.

Monotonically increasing functions “go up” when you “increase x”. Lets color that:



Example: Can you find a function f on the interval $[0, 1]$ which is bounded $|f(x)| \leq 1$ but such that $f'(x)$ is unbounded? Hint: square-”...”-beer.

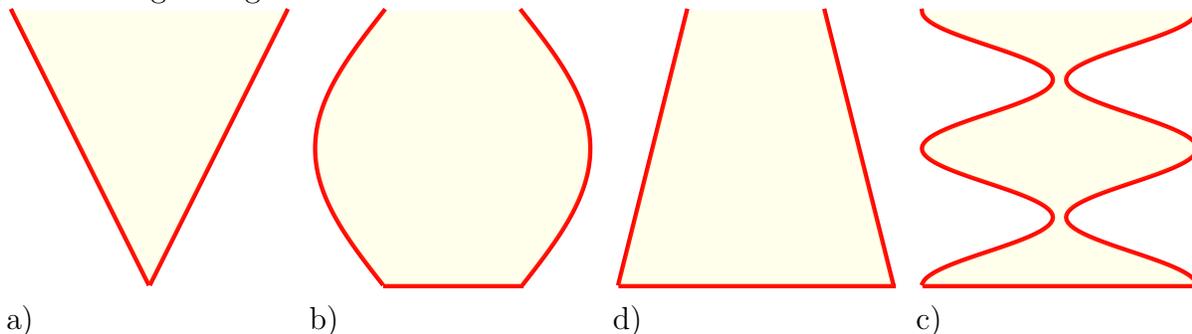
Definition: Given the function $f(x)$, we can define $g(x) = f'(x)$ and then take the derivative g' of g . This second derivative $f''(x)$ is called the **acceleration**. It measures the rate of change of the tangent slope. For $f(x) = x^4$, for example we have $f''(x) = 12x^2$. If $f''(x) > 0$ on some interval the function is called **concave up**, if $f''(x) < 0$, it is **concave down**.

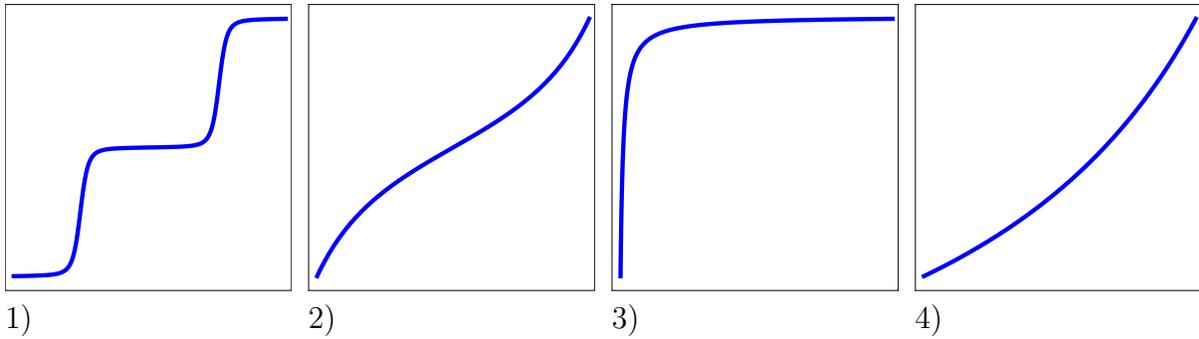
Example: Find a function f which has the property that its acceleration is constant equal to 6. **Solution.** We have to get a function such that its derivative is $6x$. That works for $3x^2$.

Example: Find a function f which has the property that its acceleration is equal to its derivative. To do so, try basic functions you know and compute $f'(x), f''(x)$ in each case.

8.6. In the famous **bottle calibration problem**, we fill a circular bottle or glass with constant amount of fluid. Plot the height of the fluid in the bottle at time t . Assume the radius of the bottle is $f(z)$ at height z . Can you find a formula for the height $g(t)$ of the water? This is not so easy. But we can find the rate of change $g'(t)$. Assume for example that f is constant, then the rate of change is constant and the height of the water increases linearly like $g(t) = t$. If the bottle gets wider, then the height of the water increases slower. There is definitely a relation between the rate of change of g and f . Before we look at this more closely, lets try to match the following cases of bottles with the graphs of the functions g qualitatively.

Example: In each of the bottles, we call g the height of the water level at time t , when filling the bottle with a constant stream of water. Can you match each bottle with the right height function?





8.7. The key is to look at $g'(t)$, the rate of change of the height function. Because $[g(t+h) - g(t)]$ times the area πf^2 is a constant times the time difference $h = dt$, we have **bottle calibration formula**

$$g' = \frac{1}{\pi f^2}.$$

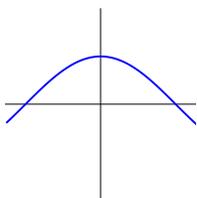
It relates the derivative function of g with the thickness $f(t)$ of the bottle at height g . No need to learn this. It just explains the story completely. It tells that that if the bottle radius f is large, then the water level increase g' is small and if the bottle radius f is small, then the liquid level change g' is large.

Homework

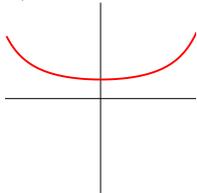
Problem 8.1: For the following functions, determine on which intervals the function is monotonically increasing or monotonically decreasing.

- a) $f(x) = -x^3 + x$ on $[-2, 2]$
- b) $f(x) = \sin(x)$ on $[-2\pi, 2\pi]$
- c) $f(x) = -x^4 + 8x^2$ on $[-4, 4]$.

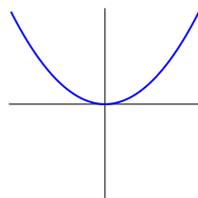
Problem 8.2: Match the following functions with their derivatives. Explain using monotonicity



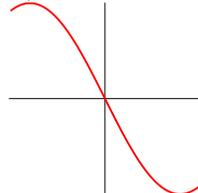
a)



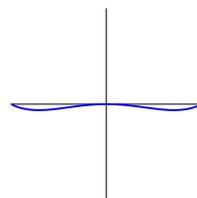
1)



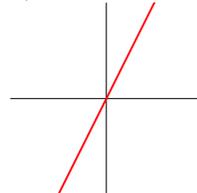
b)



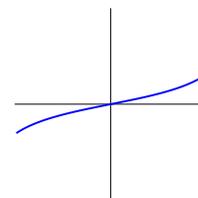
2)



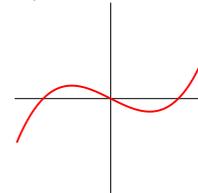
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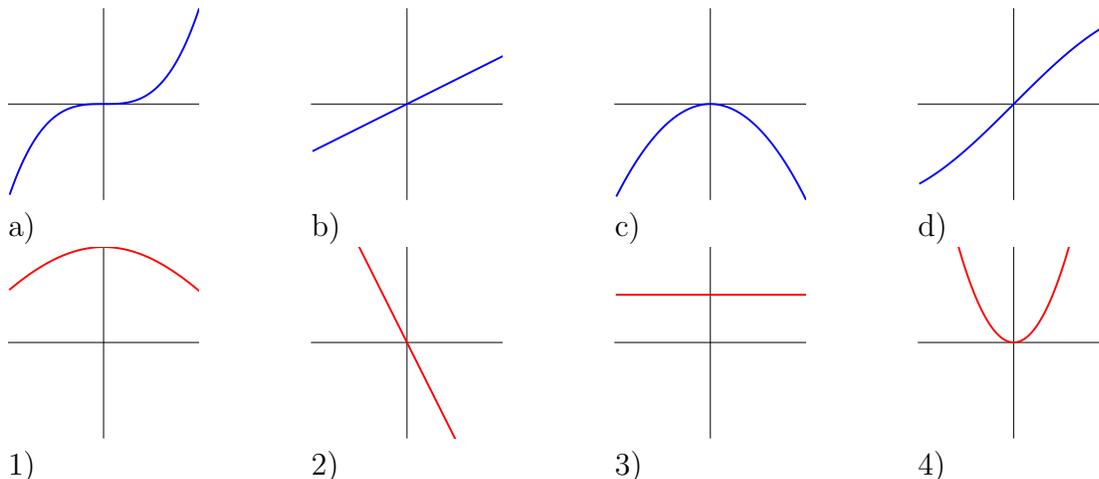


d)



4)

Problem 8.3: Match also the following functions with their derivatives. Give short explanations documenting your reasoning in each case.



Problem 8.4: Draw for the following functions the graph of the function $f(x)$ as well as the graph of its derivative $f'(x)$. You do not have to compute the derivative analytically as a formula here since we do not have all tools yet to compute the derivatives. The derivative function you draw needs to have the right qualitative shape however.

a) The **”To whom the bell tolls”** function

$$f(x) = e^{-x^2}$$

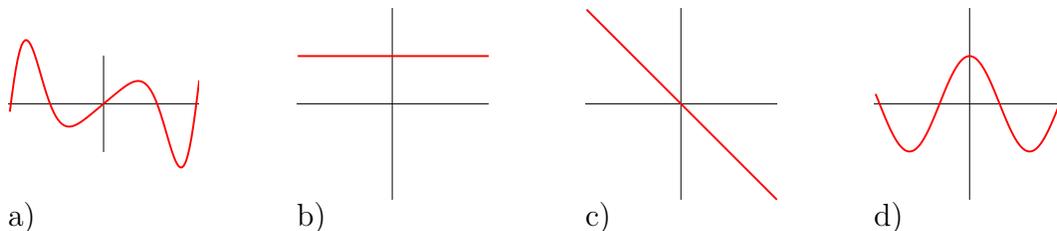
b) The **”Maria Agnesi”** function:

$$f(x) = \frac{1}{1+x^2}$$

c) The **three gorges** function

$$f(x) = \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} .$$

Problem 8.5: Below you the graphs of three derivative functions $f'(x)$. In each case you are told that $f(0) = 1$. Your task is to draw the function $f(x)$ in each of the cases a),b),c),d). Your picture does not have to be up to scale, but your drawing should display the right features.



INTRODUCTION TO CALCULUS

MATH 1A

Unit 9: Product Rule

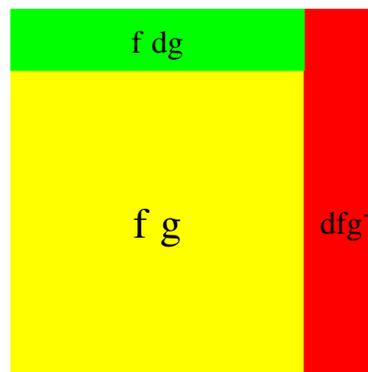
LECTURE

9.1. In this lecture, we look at the derivative of a product of functions. The product rule is also called **Leibniz rule** named after **Gottfried Leibniz**, who found it in 1684. It is important because it allows us to differentiate many more functions. We will be able to compute so the derivative of $f(x) = x \sin(x)$ for example without having to take the limit $\lim (f(x+h) - f(x))/h$. Let us start with an identity without limits. It is a discrete Leibniz rule which holds in the **Babylonian calculus** developed in the first hour.



$$f(x+h)g(x+h) - f(x)g(x) = [f(x+h) - f(x)] \cdot g(x+h) + f(x) \cdot [g(x+h) - g(x)].$$

9.2. It can be written as $D(fg) = Dfg^+ + fDg$ with a shifted function $g^+(x) = g(x+h)$ and $Df(x) = [f(x+h) - f(x)]/h$. This **quantum Leibniz rule** can also be seen geometrically: the rectangle of area $(f + df)(g + dg)$ is the union of rectangles with area $f \cdot g$, $f \cdot dg$ and $df \cdot g^+$. Now take the limit $h \rightarrow 0$:



$$\begin{aligned} \frac{[f(x+h)-f(x)]}{h} \cdot g(x+h) &\rightarrow f'(x) \cdot g(x) \\ f(x) \cdot \frac{[g(x+h)-g(x)]}{h} &\rightarrow f(x) \cdot g'(x) \end{aligned}$$

9.3. We get the extraordinarily important **product rule**:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

Example: Find the derivative function $f'(x)$ for $f(x) = x^3 \sin(x)$. **Solution:** We know how to differentiate x^3 and $\sin(x)$ so that $f'(x) = 3x^2 \sin(x) + x^3 \cos(x)$.

Example: While we know for $f(x) = x^5$ that $f'(x) = 5x^4$, let us compute this with the Leibniz rule. To do so, write $x^5 = x^4 \cdot x$. We have

$$\frac{d}{dx} x^4 = 4x^3, \quad \frac{d}{dx} x = 1.$$

The product rule gives us $f'(x) = 4x^3 \cdot x + x^4 \cdot 1 = 5x^4$. In principle we could be induction prove so the formula $f'(x^n) = nx^{n-1}$.

Example: We look now at a few derivatives related to functions, where we know the answer already but where we can check things using the product formula:

- $\frac{d}{dx}(x^3 \cdot x^5)$
- $\frac{d}{dx} e^{3x} e^{5x}$
- $\frac{d}{dx} \sqrt{x} / \sqrt{x}$
- $\frac{d}{dx} \sin(x) \cos(x)$

9.4. There is also a quotient rule which allows to differentiate $f(x)/g(x)$. Because we can write this as $f(x) \cdot 1/g(x)$, we only need to know how to differentiate $1/g(x)$. This is the **reciprocal rule**:

If $g(x) \neq 0$, then

$$\frac{d}{dx} \frac{1}{g(x)} = \frac{-g'(x)}{g(x)^2}.$$

9.5. In order to see this, write $h = 1/g$ and differentiate the equation $1 = g(x)h(x)$ on both sides. The product rule gives $0 = g'(x)h(x) + g(x)h'(x)$ so that $h'(x) = -h(x)g'(x)/g(x) = -g'(x)/g^2(x)$.

Example: Find the derivative of $f(x) = 1/x^4$. **Solution:** $f'(x) = -4x^3/x^8 = -4/x^5$. The same computation shows that $\frac{d}{dx} x^n = nx^{n-1}$ holds for all integers n .

The formula $\boxed{\frac{d}{dx} x^n = nx^{n-1}}$ holds for all integers n .

9.6. The **quotient rule** is obtained by applying the product rule to $f(x) \cdot (1/g(x))$ and using the reciprocal rule. This gives the **"Low D High take High D Low - cross the line and square the Low"** rule:

If $g(x) \neq 0$, then

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{[g(x)f'(x) - f(x)g'(x)]}{g^2(x)}.$$

Example: Find the derivative of $f(x) = \tan(x)$. **Solution:** because $\tan(x) = \sin(x)/\cos(x)$ we have

$$\tan'(x) = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}.$$

Example: Find the derivative of $f(x) = \frac{2-x}{x^2+x^4+1}$. **Solution.** We apply the quotient rule and get $[(-1)x^2 + x^4 + 1 + (2-x)(2x + 4x^3)]/(x^2 + x^4 + 1)$.

9.7. Here are some more problems with solutions:

Example: Find the second derivative of $\tan(x)$. **Solution.** We have already computed $\tan'(x) = 1/\cos^2(x)$. Differentiate this again with the quotient rule gives $-\frac{\frac{d}{dx} \cos^2(x)}{\cos^4(x)}$. We still have to find the derivative of $\cos^2(x)$. The product rule gives $\cos(x) \sin(x) + \sin(x) \cos(x) = 2 \cos(x) \sin(x)$. This gives $2 \sin(x)/\cos^3(x)$.

Example: A cylinder has volume $V = \pi r^2 h$, where r is the radius and h is the height. Assume the radius grows like $r(t) = 1 + t$ and the height shrinks like $1 - \sin(t)$. Does the volume grow or decrease at $t = 0$?

Solution: The volume $V(t) = \pi(1 + t)^2(1 - \sin(t))$ is a product of two functions $f(t) = \pi(1 + t)^2$ and $g(t) = (1 - \sin(t))$. We have $f(0) = 1, g'(0) = 2, f'(0) = 2, g(0) = 1$. The product rule gives gives $V'(0) = \pi 1 \cdot (-1) + \pi 2 \cdot 1 = \pi$. The volume increases in volume at first.

9.8. On the **Moscow papyrus** dating back to 1850 BC, the general formula $V = h(a^2 + ab + b^2)/3$ for a truncated pyramid with base length a , roof length b and height h appeared. Assume $h(t) = 1 + \sin(t), a(t) = 1 + t, b(t) = 1 - 2t$. Does the volume of the truncated pyramid grow or decrease at first? **Solution.** We could fill in $a(t), b(t), h(t)$ into the formula for V and compute the derivative using the product rule. A bit faster is to write $f(t) = a^2 + ab + b^2 = (1 + t)^2 + (1 - 3t)^2 + (1 + t)(1 - 3t)$ and note $f(0) = 3, f'(0) = -6$ then get from $h(t) = (1 + \sin(t))$ the data $h(0) = 1, h'(0) = 1$. So that $V'(0) = (h'(0)f(0) - h(0)f'(0))/3 = (1 \cdot 3 - 1(-6))/3 = -1$. The pyramid shrinks in volume at first.



Example: We pump up a balloon and let it fly. Assume that the thrust increases like t and the resistance decreases like $1/\sqrt{1-t}$ since the balloon gets smaller. The distance traveled is $f(t) = t/\sqrt{1-t}$. Find the velocity $f'(t)$ at time $t = 0$.

Homework

Problem 9.1: Find the derivatives of the following functions, then evaluate at $x = 0$

- a) $f(x) = \sin(3x) + \sin(120x) \tan(121x)$.
- b) $f(x) = \cos^4(x)/(1+x)^5$.
- c) $f(x) = e^x \sin(x) \cos(x)$.
- d) $f(x) = 3/\cos(x) + 1/\sqrt{x+1}$.
- e) $f(x) = 6xe^{5x} + 8 \tan(x)$.

Problem 9.2: a) Verify that for $f(x) = g(x)h(x)k(x)l(x)$ the formula $f' = g'hkl + gh'kl + ghk'l + ghkl'$ holds.

b) Verify the following formula for derivative of $f(x) = g(x)^4$ we have $f'(x) = 4g^3(x)g'(x)$. Do not use the chain rule but $f(x) = g(x)g(x)g(x)g(x)$ and a).

Problem 9.3: If $f(x) = \text{sinc}(x) = \sin(x)/x$, find its derivative $g(x) = f'(x)$ and then the derivative of $g(x)$. Then evaluate this at $x = 0$.

Problem 9.4: a) Find the derivative of

$$\frac{\sin(x)}{1 + \cos(x) + \frac{x^4}{\sin(x)}}$$

at $x = 0$. Try to do this as effectively as you can.

b) No gain without pain: find the derivative of

$$f(x) = \frac{1}{\sin(x) + \frac{1}{\sin(x)+1/(\sin(x)+1/\sin(x))}}$$

at $x = \pi/2$.

Problem 9.5: Here is a preparation for the chain rule, we see in the next unit. But please avoid the chain rule in a) and b) .

a) We have already computed the derivative of $g(x) = \sqrt{x}$ in the last homework. Introduce $f(x) = x^{1/4}$ and apply the product rule to $g(x) = f(x)f(x)$ to get the derivative of f .

b) Use problem 2b) applied to the identity $x = f(x)^4$ to get the derivative of f .

c) Now remember the chain rule and use it to get $f'(x)$. If you have not seen the chain rule, no problem, just look it up. We will cover it next time.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 10: Chain rule

LECTURE

10.1. If we want to take the derivative of a composition of functions like $f(x) = \sin(x^7)$, the product rule does not work. The functions are not multiplied but are “chained” in the sense that we evaluate first x^7 then apply \sin to it. In order to differentiate, we take the derivative of the x^7 then multiply this with the derivative of the function \sin evaluated at x^7 . The answer is $7x^6 \cos(x^7)$.

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) .$$

10.2.

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + (g(x+h) - g(x))) - f(g(x))]}{[g(x+h) - g(x)]} \cdot \frac{[g(x+h) - g(x)]}{h} .$$

Write $H(x) = g(x+h) - g(x)$ in the first part on the right hand side

$$\frac{f(g(x+h)) - f(g(x))}{h} = \frac{[f(g(x) + H) - f(g(x))]}{H} \cdot \frac{g(x+h) - g(x)}{h} .$$

As $h \rightarrow 0$, we also have $H \rightarrow 0$ and the first part goes to $f'(g(x))$ and the second factor has $g'(x)$ as a limit.

10.3. Let us look at some examples.

Example: Find the derivative of $f(x) = (4x^2 - 1)^{17}$. **Solution** The inner function is $g(x) = 4x^2 - 1$ with derivative $8x$. We get therefore $f'(x) = 17(4x^2 - 1)^{16} \cdot 8x$. Remark. We could have expanded out the power $(4x^2 - 1)^{17}$ first and avoided the chain rule. Try it. You will see that the rule of avoiding the **chain rule** is called the **pain rule**.

Example: Find the derivative of $f(x) = \sin(\pi \cos(x))$ at $x = 0$. **Solution:** applying the chain rule gives $\cos(\pi \cos(x)) \cdot (-\pi \sin(x))$.

Example: For linear functions $f(x) = ax + b, g(x) = cx + d$, the chain rule can readily be checked: we have $f(g(x)) = a(cx + d) + b = acx + ad + b$ which has the derivative ac . This agrees with the definition of f times the derivative of g . You can convince you that the chain rule is true also from this example since if you look closely at a point, then the function is close to linear.

10.4. One of the cool applications of the chain rule is that we can compute derivatives of inverse functions:

Example: Find the derivative of the natural logarithm function $\log(x)$.¹ **Solution** Differentiate the identity $\exp(\log(x)) = x$. On the right hand side we have 1. On the left hand side the chain rule gives $\exp(\log(x)) \log'(x) = x \log'(x) = 1$. Therefore $\log'(x) = 1/x$.

$$\frac{d}{dx} \log(x) = 1/x.$$

Definition: Denote by $\arccos(x)$ the inverse of $\cos(x)$ on $[0, \pi]$ and with $\arcsin(x)$ the inverse of $\sin(x)$ on $[-\pi/2, \pi/2]$.

Example: Find the derivative of $\arcsin(x)$. **Solution.** We write $x = \sin(\arcsin(x))$ and differentiate.

$$\frac{d}{dx} \arcsin(x) = \frac{1}{\sqrt{1-x^2}}.$$

Example: Find the derivative of $\arccos(x)$. **Solution.** We write $x = \cos(\arccos(x))$ and differentiate.

$$\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1-x^2}}.$$

Example: $f(x) = \sin(x^2 + 3)$. Then $f'(x) = \cos(x^2 + 3)2x$.

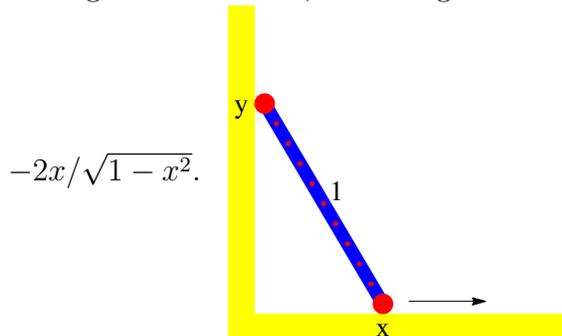
Example: $f(x) = \sin(\sin(\sin(x)))$. Then $f'(x) = \cos(\sin(\sin(x))) \cos(\sin(x)) \cos(x)$. Why is the chain rule called "chain rule". The reason is that we can chain even more functions together.

Example: Let us compute the derivative of $\sin(\sqrt{x^5 - 1})$ for example. **Solution:** This is a composition of three functions $f(g(h(x)))$, where $h(x) = x^5 - 1$, $g(x) = \sqrt{x}$ and $f(x) = \sin(x)$. The chain rule applied to the function $\sin(x)$ and $\sqrt{x^5 - 1}$ gives $\cos(\sqrt{x^5 - 1}) \frac{d}{dx} \sqrt{x^5 - 1}$. Apply now the chain rule again for the derivative on the right hand side.

¹We always write $\log(x)$ for the natural log. Similarly as $\exp(x) = e^x$, one can also use \ln which stands for "logarithmus naturalis". Practically all computer languages like Python, C, Perl, R, Matlab, Mathematica use \log . Paul Halmos called "ln" a childish notation which no mathematician ever used.

Example: Here is a famous **falling ladder problem**.

A stick of length 1 slides down a wall. How fast does it hit the floor if it slides horizontally on the floor with constant speed? The ladder connects the point $(0, y)$ on the wall with $(x, 0)$ on the floor. We want to express y as a function of x . We have $y = f(x) = \sqrt{1 - x^2}$. Taking the derivative, assuming $x' = 1$ gives $f'(x) =$



In reality, the ladder breaks away from the wall. One can calculate the force of the ladder to the wall. The force becomes zero at the **break-away angle** $\theta = \arcsin((2v^2/(3g))^{2/3})$, where g is the gravitational acceleration and $v = x'$ is the velocity.

Example: For the brave: find the derivative of $f(x) = \cos(\cos(\cos(\cos(\cos(\cos(\cos(x)))))$)).

Example: Take the derivative of $f_3(x) = e^{e^x}$.

Solution We can also write this as $\exp(\exp(\exp(x)))$. The derivative is

$$\exp(\exp(\exp(x))) \exp(\exp(x)) \exp(x) .$$

Example: Lets push that to the extreme and differentiate

$$f(x) = \exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(\exp(x))$$

Here is the poetic formula obtained when running this in Mathematica:

$$F[f.] := \text{Exp}[f]; \text{D}[\text{Last}[\text{NestList}[F, x, 11]], x]$$

$$\exp \left(e^{e^{e^{e^{e^{e^{e^{e^{e^{e^{e^x}}}}}}}} + e^{e^{e^{e^{e^{e^{e^{e^{e^{e^x}}}}} + e^{e^{e^{e^{e^{e^{e^{e^{e^x}}}}} + e^{e^{e^{e^{e^{e^{e^{e^x}}}}} + e^{e^{e^{e^{e^{e^{e^x}}}}} + e^{e^{e^{e^{e^{e^x}}}}} + e^{e^{e^{e^x}}} + e^{e^{e^x}} + e^{e^x} + e^x + x \right)$$

Example: Find the derivative of $1/\sin(x)$ using the quotient rule.

Solution $-\cos(x) \cdot 1/\sin^2(x)$.

Example: Find the derivative of $f(x) = 1/\sin(x)$ using the chain rule.

Solution. The outer function is $f(x) = 1/x$. Therefore $f'(x) = -\cos(x)/\sin^2(x)$.

Homework

Problem 10.1: Find the derivatives of the following functions:

a) $f(x) = \sin(\log(x))$

c) $f(x) = \exp(1/(1+x^2))$

b) $f(x) = \tan(x^{11})$

d) $(3 + \sin(x))^{-5}$

Problem 10.2: Find the derivatives of the following functions at $x = 1$.

a) $f(x) = -x \log(x)$. (where \log is natural log)

c) $(1 + x^2 + x^4)^{100}$

b) $\sqrt{x^5 + 1}$

d) $\frac{5x^4}{2\sqrt{x^5+1}}$

Problem 10.3: a) Find the derivative of $f(x) = 1/x$ by differentiating the identity $xf(x) = 1$ and using the product rule.

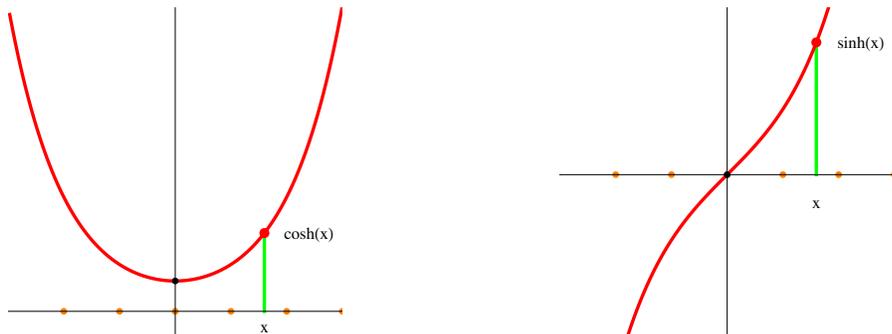
b) Find the derivative of $f(x) = \operatorname{arccot}(x)$ by differentiating $\cot(\operatorname{arccot}(x)) = x$ and using the chain rule.

Problem 10.4: a) Find the derivative of \sqrt{x} by differentiating the identity $f(x)^2 = x$ leaving f as it is and solving for $f'(x)$.

b) Find the derivative of $x^{m/n}$ by differentiating the identity $f(x)^n = x^m$ leaving f as it is and solving for $f'(x)$.

Problem 10.5: a) Find the derivative of the inverse $\operatorname{arccosh}(x)$ of $\cosh(x)$ by using the chain rule.

b) Find the derivative of the inverse $\operatorname{arsinh}(x)$ of $\sinh(x)$ by using the chain rule.



Define $\cosh(x) = [\exp(x) + \exp(-x)]/2$ and $\sinh(x) = [\exp(x) - \exp(-x)]/2$. the **hyperbolic cosine** and **hyperbolic sine**. The cosh function is the shape of a chain hanging at two points. The shape is the hyperbolic cosine. You check $\cosh^2(x) - \sinh^2(x) = 1$.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 11: Critical Points

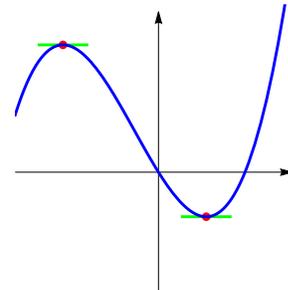
LECTURE

11.1. An important goal of life is to maximize nice quantities and minimize unpleasant ones. Extremizing quantities is also a most important principle which nature follows: laws in physics like Newton's law, or the Maxwell equations describing light, or equations describing matter can be based on the principle of extremization. The most important intuitive insight is that at maxima or minima of a function f , the tangent to the graph must be horizontal. This leads to the following notion for differentiable functions:

Definition: A point x_0 is a **critical point** of f if $f'(x_0) = 0$.

11.2. In some textbooks, critical points also include points, where f' is not defined. Others also include boundary points.¹ So, we do here **not** include boundary points in the list of critical points. These points are considered to be outside the domain of definition of f' and deal with them separately.

Example: Find the critical points of the function $f(x) = x^3 + 3x^2 - 24x$. **Solution:** we compute the derivative as $f'(x) = 3x^2 + 6x - 24$. The roots of f' are 2, -4.



Definition: A point is called a **local maximum** of f , if there exists an interval $U = (p - a, p + a)$ around p , such that $f(p) \geq f(x)$ for all $x \in U$. A **local minimum** is a local maximum of $-f$. Local maxima and minima together are called **local extrema**.

Example: The point $x = 0$ is a local maximum for $f(x) = \cos(x)$. The reason is that $f(0) = 1$ and $f(x) < 1$ nearby.

Example: The point $x = 1$ is a local minimum for $f(x) = (x - 1)^2$. The function is zero at $x = 1$ and positive everywhere else.

¹In all more advanced math textbooks, critical points are defined as such. Important definitions have to be simple.

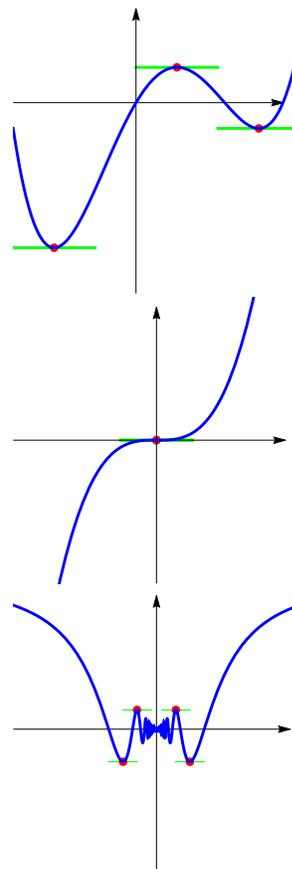
Fermat: If f is differentiable and has a local extremum at x , then $f'(x) = 0$.

11.3. Why is this so? Assume the derivative $f'(x) = c$ is non-zero. We can assume $c > 0$ otherwise replace f with $-f$. By the definition of limits, for some small enough h , we have $f(x+h) - f(x)/h \geq c/2$. But this means $f(x+h) \geq f(x) + hc/2$ and x can not be a local maximum.

Example: The derivative of $f(x) = 72x - 30x^2 - 8x^3 + 3x^4$ is $f'(x) = 72 - 60x - 24x^2 + 12x^3$. By plugging in integers (calculus teachers like integer roots because students like integer roots!) we can guess the roots $x = 1, x = 3, x = -2$ and see $f'(x) = 12(x-1)(x+2)(x-3)$. The critical points are $1, 3, -2$.

Example: We have already seen that $f'(x) = 0$ does not imply that x is a local maximum or minimum. The function $f(x) = x^3$ is a counter example. It satisfies $f'(0) = 0$ but 0 is neither a minimum nor maximum there. It is an example of an **inflection point**, which is a point, where the second derivative f'' changes sign.

Example: The function $f(x) = x \sin(1/x)$ is continuous at $x = 0$ but there are infinitely many critical points near 0 . The function f is not differentiable at 0 , the derivative $\sin(1/x) - \cos(1/x)/x$ not only oscillates like crazy at $x = 0$, it also blows up at $x = 0$.



11.4. If $f''(x) > 0$, then the graph of the function is **concave up**. If $f''(x) < 0$ then the graph of the function is **concave down**.

Second derivative test. If x is a critical point of f and $f''(x) > 0$, then f is a local minimum. If $f''(x) < 0$, then f is a local maximum.

11.5. If $f''(x_0) > 0$ then $f'(x)$ is negative for $x < x_0$ and positive for $f'(x) > x_0$. This means that the function decreases left from the critical point and increases right from the critical point. The point x_0 is a local minimum. Similarly, if $f''(x_0) < 0$ then $f'(x)$ is positive for $x < x_0$ and $f'(x)$ is negative for $x > x_0$. This means that the function increases left from the critical point and decreases right from the critical point. The point is a local maximum.

Example: The function $f(x) = x^2$ has one critical point at $x = 0$. Its second derivative is 2 there.

Example: Find the local maxima and minima of the function $f(x) = x^3 - 3x$ using the second derivative test. **Solution:** $f'(x) = 3x^2 - 3$ has the roots $1, -1$. The second

derivative $f''(x) = 6x$ is negative at $x = -1$ and positive at $x = 1$. The point $x = -1$ is therefore a local maximum and the point $x = 1$ is a local minimum.

Example: Find the local maxima and minima of the function $f(x) = \cos(\pi x)$ using the second derivative test.

Example: For the function $f(x) = x^6 - x^4$, the second derivative test is inconclusive at $x = 0$. Can you nevertheless see what the nature of the critical point 0 is?

Example: Also for the function $f(x) = x^4$, the second derivative test is inconclusive at $x = 0$. The second derivative is zero. Can you nevertheless see whether the critical point 0 is a local maximum or a local minimum?

Let us look at an example, where we can review the chain rule.

Example: Find the critical points of $f(x) = 4 \arctan(x) + x^2$. **Solution.** The derivative is

$$f'(x) = \frac{4}{1+x^2} + 2x = \frac{2x + 2x^3 + 4}{1+x^2}.$$

We see that $x = -1$ is a critical point. There are no other roots of $2x + 2x^3 + 4 = 0$. How did we get the derivative of \arctan again? Differentiate: $\tan(\arctan(x)) = x$ and write $u = \arctan(x)$:

$$\frac{1}{\cos^2(u)} \arctan'(x) = 1.$$

Use the identity $1 + \tan^2(u) = 1/\cos^2(u)$ to write this as

$$(1 + \tan^2(u)) \arctan'(x) = 1.$$

But $\tan(u) = \tan(\arctan(x)) = x$ so that $\tan^2(u) = x^2$. And we have $(1+x^2) \arctan'(x) = 1$. Solving for $\arctan'(x)$ gives $\arctan'(x) = \frac{1}{1+x^2}$.

Example: Finally, let us look at the function $\cosh(x) = (e^x + e^{-x})/2$. We have seen this function already in the last homework. It is the chain curve, because its graph is the shape of a chain. Its sibling is $\sinh(x)$, spelled “sinch” like “grinch”. The cosh function was viral a few days ago because of the hanging cable problem.

Can You Solve the 'Hanging Cable' Problem, Used as an Amazon Interview Question?

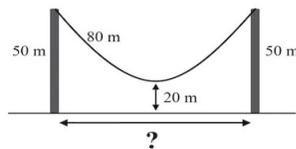


Posted by msmash on Wednesday February 19, 2020 @12:27PM from the how-about-that dept.

An anonymous reader [shares a problem that Amazon asks in its interviews:](#)

A cable of 80 meters is hanging from the top of two poles that are both 50 meters off the ground. What is the distance between the two poles (to one decimal point) if the center cable is (a) 20 meters off the ground and (b) 10 meters off the ground?

(a) 20 m above ground



Homework

Problem 11.1: Find all critical points for the following functions. If there are infinitely many, indicate their structure. For $f(x) = \cos(x)$ for example, the critical points can be written as $\pi/2 + k\pi$, where k is an integer.

- a) $f(x) = x^6 - 3x^2$.
- b) $f(x) = 4\sin(\pi x) + 3$
- c) $f(x) = \exp(-x^2)x^2$.
- d) $f(x) = \sin(\cos(\pi x))$

Problem 11.2: Find all the critical points and use the second derivative test to determine whether they are maxima or minima.

- a) $f(x) = x \log(x)$, where $x > 0$.
- b) $f(x) = 1/(1 + x^2)$
- c) $f(x) = x^2 - 2x + 1$.
- d) $f(x) = 2x \tan(x)$, where $-\pi/2 < x < \pi/2$

Problem 11.3: a) Verify that any cubic equation $f(x) = x^3 + ax^2 + bx + c$ always has an inflection point, a point where $f''(x)$ changes sign.
b) Where is the inflection point for $a = 2, b = 1, c = 3$? Is this point also a critical point? Is it a maximum or minimum?

Problem 11.4: Depending on c , the function $f(x) = x^4 - cx^2$ has either one or three critical points. Use the second derivative test to decide: a) For $c = 1$, find and determine the nature of the critical points.
b) For $c = -1$, find and determine the nature of the critical points.

Problem 11.5: a) Find your own function which has exactly one local maximum and local minimum on the real line.
b) Engineer a concrete function which has exactly 2 local maximum and 1 local minimum.
c) Find a differentiable function on the real line with 2 local maxima and no local minimum. If it works, give one. If it does not work, give a reason why it does not work.
d) Can you draw a function which is not everywhere differentiable for which we have 2 local maxima and no local minimum?

INTRODUCTION TO CALCULUS

MATH 1A

Unit 12: Global extrema

LECTURE

12.1. In this lecture we are interested in the points where a function is maximal overall. These **global extrema** can occur at critical points of f or at the boundary of the domain, where f is defined.

Definition: A point a is called a **global maximum** of f if $f(a) \geq f(x)$ for all x . A point a is called a **global minimum** of f if $f(a) \leq f(x)$ for all x .

12.2. How do we find global maxima? The answer is simple: make a list of all local extrema and boundary points, then pick the largest. Global maxima or minima do not need to exist however. The function $f(x) = x^2$ has a global minimum at $x = 0$ but no global maximum. The function $f(x) = x^3$ has no global maximum nor minimum at all. We can however look at global maxima on finite intervals.

Example: Let us look at the example from last week where we found the square of maximal area among all squares of side length $x, 1 - x$. The function $f(x) = x(1 - x)$ had a maximum at $x = 1/2$. We also have to look at the boundary points. Why? Because both x and $f(x)$ can not become negative. We see that $f(x)$ has to be looked at on the interval $[0, 1]$. To decide about global maxima, just look at the critical points and boundary points and pick the maximal.

Example: Find the global maximum of $f(x) = x^2$ on the interval $[-1, 2]$. **Solution.** We look for local extrema at critical points and at the boundary. Then we compare all these extrema to find the maximum or minimum. The critical points are $x = 0$. The boundary points are $-1, 2$. Comparing the values $f(-1) = 1, f(0) = 0$ and $f(2) = 4$ shows that f has a global maximum at 2 and a global minimum at 0.

Extreme value theorem of Bolzano: A continuous function f on a finite interval $[a, b]$ attains a global maximum and a global minimum.

Proof: for every n , make a list of the points $x_k = (a + (k/n)(b - a))$ where $k = 1, \dots, n$. Pick the one where $f(x_k)$ is maximal one and call this x_n . Now we use the **Bolzano-Weierstrass theorem** which assures that any sequence of numbers on a closed interval $[a, b]$ has an accumulation point. Such an accumulation point is a maximum. Similarly, we can construct the minimum.

The **Bolzano-Weierstrass theorem** is verified constructively too: cut the interval in

two equal parts and choose a part which contains infinitely many points x_n . We have reduced the problem to a smaller interval. Now take this interval and again divide it into two. Again chose the one in which x_n hits infinitely many times. Wash, rinse and repeat this again and again leads to smaller and smaller intervals of size $[b - a]/2^n$ in which there are infinitely many points. Note that these intervals are nested so that they lead to a limit (if the interval were $[0, 1]$ and we would cut each time into 10 pieces, then we would gain in every step one digit of the decimal expansion of the number we are looking for).

12.3. Note that the global maximum or minimum can also also on the boundary or points where the derivative does not exist.

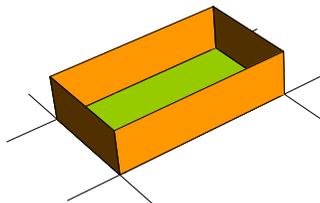
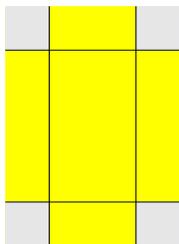
Example: Find the global maximum and minimum of the function $f(x) = |x|$. The function has no absolute maximum as it goes to infinity for $x \rightarrow \infty$. The function has a global minimum at $x = 0$ but the function is not differentiable there. The point $x = 0$ is a point which does not belong to the domain of f' .

Example: A **soda can** is a cylinder of volume $\pi r^2 h$. The surface area $2\pi r h + 2\pi r^2$ measures the amount of material used to manufacture the can. Assume the surface area is 2π , we can solve the equation for $h = (1 - r^2)/r = 1/r - r$ **Solution:** The volume is $f(r) = \pi(r - r^3)$. Find the can with maximal volume: $f'(r) = \pi - 3r^2\pi = 0$ showing $r = 1/\sqrt{3}$. This leads to $h = 2/\sqrt{3}$.



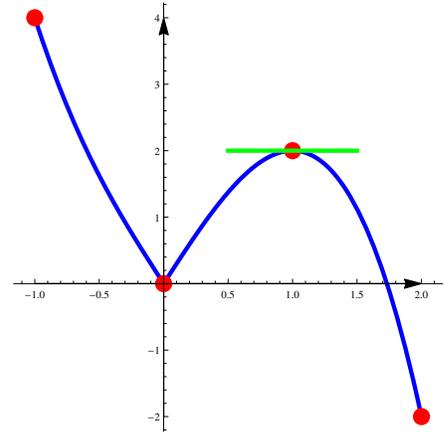
Example: Take a card of 2×2 inches. If we cut out 4 squares of equal side length x at the corners, we can fold up the paper to a tray with width $(2 - 2x)$ length $(2 - 2x)$ and height x . For which $x \in [0, 1]$ is the tray volume maximal?

Solution The volume is $f(x) = (2 - 2x)(2 - 2x)x$. To find the maximum, we need to compare the critical points which is at $x = 1/3$ and the boundary points $x = 0$ and $x = 1$.



Example: Find the global maxima and minima of the function $f(x) = 3|x| - x^3$ on the interval $[-1, 2]$.

Solution. For $x > 0$ the function is $3x - x^3$ which can be differentiated. The derivative $3 - 3x^2$ is zero at $x = 1$. For $x < 0$ the function is $-3x - x^3$. The derivative is $-3 - x^2$ and has no root. The only critical points are 1. There is also the point $x = 0$ which is not in the domain where we can differentiate the function. We have to deal with this point separately. We also have to look at the boundary points $x = -1$ and $x = 2$. Making a list of function values at $x = -1, x = 0, x = 1, x = 2$ gives the maximum.



Homework

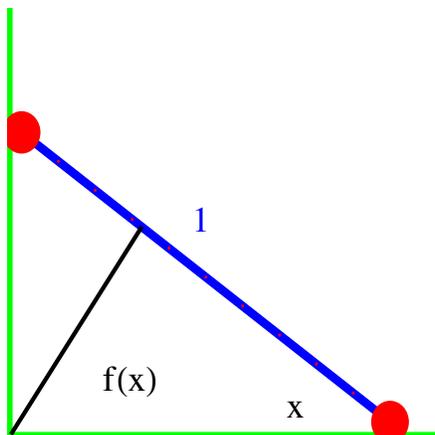
Problem 1: Find all the local maxima and minima as well as the global maximum and the global minimum of the function $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x$ on the closed interval $[-2, 3]$. Make sure to compute the critical points inside the interval and then compare also the boundary points.

Problem 2: Find the global maximum and minimum of the function $f(x) = 2x^3 - 3x^2 - 36x$ on the interval $[-4, 4]$

Problem 3: Mathcandy.com (look it up!) manufactures spherical candies. Its effectiveness is $A(r) - V(r)$, where $A(r)$ is the surface area and $V(r)$ the volume of a candy of radius r . Find the radius, where $f(r) = A(r) - V(r)$ has a global maximum for $r \geq 0$.



Problem 4: A ladder of length 1 is one side at a wall and on one side at the floor. a) Verify that the distance from the ladder to the corner is $f(x) = \sin(x) \cos(x)$. b) Find the angle x for which $f(x)$ is maximal.



Problem 5: The function $S(x) = -x \log(x)$ is called the **entropy function**. Find the probability $0 < x \leq 1$ which maximizes entropy. One of the most important principles in science is that nature tries to maximize entropy. In some sense we compute here the number of maximal entropy.

Entropy has been introduced by Ludwig Boltzmann. It is important in physics and chemistry. $S = k \log(W)$ is interpreted as $W = 1/p$, then take the expectation giving $S = -k \sum_p p \log(p)$. Note that log and not ln is used.



INTRODUCTION TO CALCULUS

MATH 1A

Unit 13: Hospital's rule

LECTURE

13.1. As an other application of calculus, we look at Hospital's rule. ¹ It is a miracle procedure which resolves all worries about limits:

Hospital's rule. If f, g are differentiable and $f(p) = g(p) = 0$ and $g'(p) \neq 0$, then

$$\lim_{x \rightarrow p} \frac{f(x)}{g(x)} = \lim_{x \rightarrow p} \frac{f'(x)}{g'(x)} .$$

Lets see how it works in examples:

Example: Lets prove **the fundamental theorem of trigonometry** again:

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = \lim_{x \rightarrow 0} \frac{\cos(x)}{1} = 1 .$$

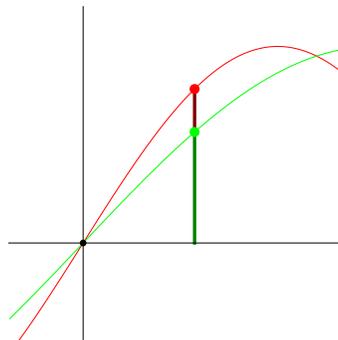
Why did we work so hard for this? We used the fundamental theorem to derive the derivatives for cos and sin at all points. In order to apply l'Hospital, we had to know the derivative. Our work to establish the limit was not in vain.

13.2. The proof of the rule is almost comic in its simplicity. Especially after we will see how fantastically useful it is:

since $f(p) = g(p) = 0$ we have $Df(p) = f(p+h)/h$ and $Dg(p) = g(p+h)/h$ so that for every $h > 0$ with $g(p+h) \neq 0$ the **quantum l'Hospital rule** holds:

$$\frac{f(p+h)}{g(p+h)} = \frac{Df(p)}{Dg(p)} .$$

Now take the limit $h \rightarrow 0$. Voilà!



¹Also Hôpital. Hospital is is easier to write and remember (bring f to the hospital!)

13.3. Sometimes, we have to administer a medicine twice. To use this, l'Hospital can be improved in that the condition $g'(0) = 0$ can be replaced by the requirement that the limit $\lim_{x \rightarrow p} f'(x)/g'(x)$ exists. Instead of having a rule which replaces a limit with an other limit and cure a disease with a new one, we formulate it how it is used. The second derivative case could easily be generalized for higher derivatives. There is no need to memorize this. Just remember that you can check in several times to a hospital.

If $f(p) = g(p) = f'(p) = g'(p) = 0$ then $\lim_{x \rightarrow p} \frac{f(x)}{g(x)} = \lim_{x \rightarrow p} \frac{f''(x)}{g''(x)}$ if the limit to the right exists.

Example: Find the limit $\lim_{x \rightarrow 0} (1 - \cos(x))/x^2$. This limit had been pivotal to compute the derivatives of trigonometric functions. **Solution:** differentiation gives

$$\lim_{x \rightarrow 0} -\sin(x)/2x .$$

Now apply l'Hospital again.

$$\lim_{x \rightarrow 0} -\sin(x)/(2x) = \lim_{x \rightarrow 0} -\cos(x)/2 = -\frac{1}{2} .$$

Example: Problem. Find the limit $f(x) = (\exp(x^2) - 1)/\sin(x^2)$ for $x \rightarrow 0$.

Example: Problem: What do you get if you apply l'Hospital to the limit $[f(x+h) - f(x)]/h$ as $h \rightarrow 0$?

Answer: Differentiate both sides with respect to h ! And then feel awesome!

Example: Find $\lim_{x \rightarrow \infty} x \sin(1/x)$. **Solution.** Write $y = 1/x$ then $\sin(y)/y$. Now we have a limit, where the denominator and nominator both go to zero.

The case when both sides converge to infinity can be reduced to the $0/0$ case by looking at $A = f/g = (1/g(x))/(1/f(x))$ which has the limit $g'(x)/g^2(x)/f'(x)/f^2(x) = g'(x)/f'(x)((1/g)/(1/f))^2 = g'/f'(f^2/g^2) = (g'/f')A^2$, so that $A = f'(p)/g'(p)$. We see:

If $\lim_{x \rightarrow p} f(x) = \lim_{x \rightarrow p} g(x) = \infty$ for $x \rightarrow p$ and $g'(p) \neq 0$, then

$$\lim_{x \rightarrow p} \frac{f(x)}{g(x)} = \frac{f'(p)}{g'(p)} .$$

Example: What is the limit $\lim_{x \rightarrow 0} x^x$? This will provide the best answer to the question **What is 0^0 ?**

Solution: Because $x^x = e^{x \log(x)}$, it is enough to understand the limit $x \log(x)$ for $x \rightarrow 0$.

$$\lim_{x \rightarrow 0} \frac{\log(x)}{1/x} .$$

Now the limit can be seen as the limit $(1/x)/(-1/x^2) = -x$ which goes to 0. Therefore $\lim_{x \rightarrow 0} x^x = 1$. (We assume that $x > 0$ in order to have real values x^x . If we want a function defined everywhere take $|x|^{|x|}$.)

Example: Find the limit $\lim_{x \rightarrow 2} \frac{x^2 - 4x + 4}{\sin^2(x-2)}$.

Solution: this is a case where $f(2) = f'(2) = g(2) = g'(2) = 0$ but $g''(2) = 2$. The limit is $f''(2)/g''(2) = 2/2 = 1$.

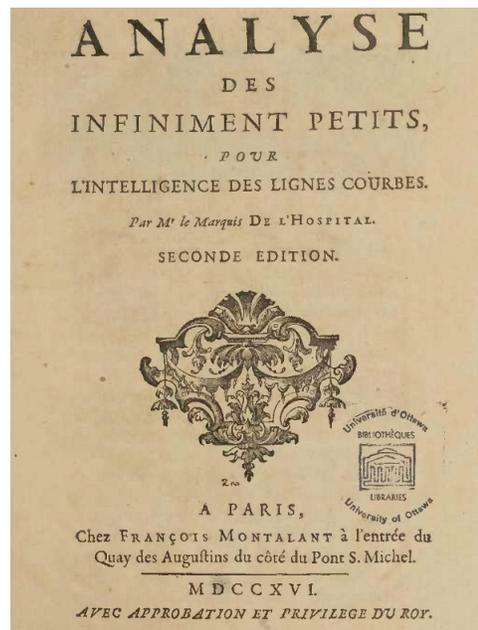
13.4. Hospital's rule always works in calculus situations, where functions are differentiable. The rule can fail if differentiability of f or g fails. Here is an other "rare" example, where one has to think a bit more:

Example: Deja Vue: Find $\frac{\sqrt{x^2+1}}{x}$ for $x \rightarrow \infty$. L'Hospital gives $x/\sqrt{x^2+1}$ which in terms gives again $\frac{\sqrt{x^2+1}}{x}$. Apply l'Hospital again to get the original function. We got an infinite loop. If the limit is A , then the procedure tells that it is equal to $1/A$. The limit must therefore be 1. This case can be covered easily without going to the hospital: divide both sides by x to get $\sqrt{1+1/x^2}$. Now, we can see the limit 1.

Example: Trouble? The limit $\lim_{x \rightarrow \infty} (2x + \sin(x))/3x$ is clearly $2/3$ since we can take the sum apart and have $2/3 + \sin(x)/(3x)$. Hospital gives $\lim_{x \rightarrow \infty} (2 + \cos(x))/3$ which has no limit. This is not trouble, since Hospital applies only if the limit $f'(x)$ and $g'(x)$ exists.

History

13.5. The "first calculus book", the world has known was "Analyse des Infiniment Petits pour l'intelligence des Lignes Courbes". It appeared in 1696 and was written by **Guillaume de l'Hospital**, a text if typeset in a modern font would probably fit onto 50-100 pages.² It is now clear that the mathematical content in Hospital's book is mostly due to **Johannes Bernoulli**. The book remained the standard for calculus textbooks for a century.



²Stewart's book with 1200 pages probably contains about 4 million characters, about 12 times more than l'Hospital's book. Modern calculus books also contain more material of course. The OCR text of l'Hospital's book of 200 pages has 300'000 characters.

Homework

Problem 13.1: For the following functions, find the limits as $x \rightarrow 0$:

- a) $9x/\sin(3x)$
- b) $(\exp(6x) - 1)/(\exp(7x) - 1)$
- c) $\sin^2(4x)/\sin^2(5x)$
- d) $\frac{\sin(x^2)}{\sin^2(x)}$
- e) $\sin(\sin(11x))/x$.

Problem 13.2: For the following functions, find the limits as $x \rightarrow 1$:

- a) $\sin(\pi x)/x$.
- b) $(x - 1)^2/(\cos(x - 1) - 1)$
- c) $(\exp(x) - e)/(\exp(3x) - e^3)$
- d) $(x - 4)/(4x + \sin(\pi x))$
- e) Find $\lim_{x \rightarrow \infty} (x^2 + x - 1)/\sqrt{4x^4 + 1}$.
(**Hint.** Find the limit of $(x^2 + x - 1)^2/(4x^4 + 1)$ first, then take the square root of the limit. Apply Hospital several times).

Problem 13.3: Use l'Hospital to compute the following limits at $x = 0$:

- a) $\log |5x|/\log |x|$.
- b) $\lim_{x \rightarrow 0} x/\log |x|$
- c) $4\text{sinc}'(x) = 4(\cos(x)x - \sin(x))/x^2$
- d) $\log |1 + x|/\log |\log |1 + x||$.
- e) $(e^x - 1)/(e^{2x} - 1)$

Problem 13.4: We have seen how to compute limits with healing. Solve the following healing problems with l'Hospital at $x = 1$:

- a) $\frac{x^{1000}-1}{x^{40}-1}$.
- b) $\frac{\tan^2(x-1)}{(\cos(x-1)-1)}$

Problem 13.5: More practice.

- a) Find the limit $\lim_{x \rightarrow 5} \frac{x^2-25}{x-5}$
- b) Find the limit $\lim_{x \rightarrow 0} \frac{1-e^x}{x-x^3}$.
- c) Find the limit $\lim_{x \rightarrow 0} \frac{\log(1+9x)}{4x}$.
- d) Find the limit $\lim_{x \rightarrow 1} (x^7 - 1)/(x^3 - 1)$.
- e) Find the limit $\lim_{x \rightarrow 0} \frac{13x}{\tan(6x)}$.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 14: Newton method

LECTURE

14.1. In the lecture on the intermediate value theorem, we found roots of functions using a “divide and conquer” technique: start with an interval $[a, b]$ for which $f(a) < 0$ and $f(b) > 0$. If $f((a + b)/2)$ is positive, then use the interval $[a, (a + b)/2]$ otherwise $[(a + b)/2, b]$. After n steps, we are $(b - a)/2^n$ close to the root. If the function f is differentiable, we can do better and use the value of the derivative to get closer to a point $y = T(x)$. Lets find this point y . If we draw a tangent at $(x, f(x))$ and intersect it with the x -axes, then

$$f'(x) = \frac{f(x) - 0}{x - T(x)}.$$

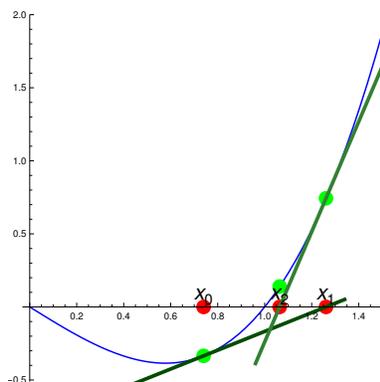
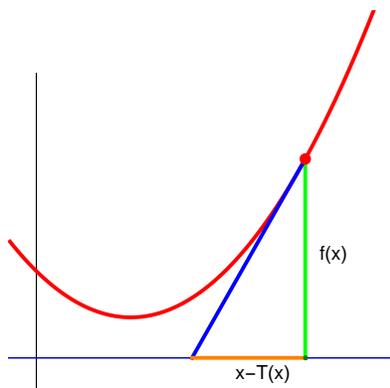
Now, $f'(x)$ is the slope of the tangent and the right hand side is ”rise” over ”run” (see the picture). If we solve for $T(x)$, we get

Definition: The **Newton map** is defined as

$$T(x) = x - \frac{f(x)}{f'(x)}.$$

14.2. The **Newton’s method** applies this map a couple of times until we are sufficiently close to the root: start with a point x , then compute a new point $x_1 = T(x)$, then $x_2 = T(x_1)$ etc.

If p is a root of f such that $f'(p) \neq 0$, and x_0 is close enough to p , then $x_1 = T(x_0)$, $x_2 = T(x_1)$ converges to the root p .



Example: If $f(x) = ax + b$, we reach the root in one step.

Example: If $f(x) = x^2$ then $T(x) = x - x^2/(2x) = x/2$. We get exponentially fast to the root 0. In general, the method is much better:

The Newton method converges extremely fast to a root $f(p) = 0$ if $f'(p) \neq 0$. In general, the number of correct digits double in each step.

In 4 steps we expect to have $2^4 = 16$ digits correct. Having a fast method to compute roots is useful. For example, in computer graphics, where things can not be fast enough. We will explore a bit in the lecture how fast the method is.

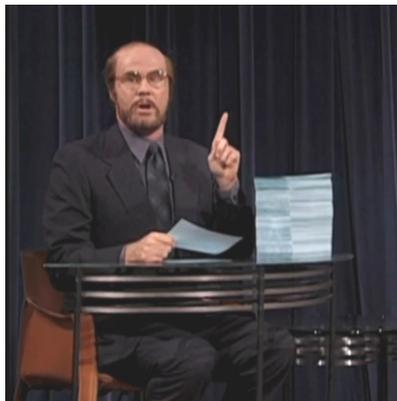
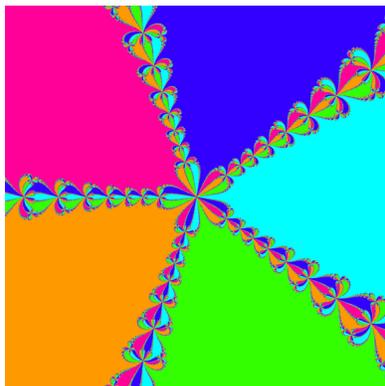
14.3. If we have several roots, and we start at some point, to which root will the Newton method converge? Does it at all converge? This is an interesting question. It is also historically intriguing because it is one of the first cases, where "chaos" was observed at the end of the 19'th century.

Example: Find the Newton map in the case $f(x) = x^5 - 1$.

Solution

$$T(x) = x - \frac{x^5 - 1}{5x^4}.$$

14.4. If we look for roots in the complex like for $f(x) = x^5 - 1$ which has 5 roots in the complex plane, the "region of attraction" of each of the roots is a complicated set which we call the **Newton fractal**.



The Newton method is scumtrulescent! (To quote from the "Inside the Actors Studio" at SNL)

Example: Lets compute $\sqrt{2}$ to 12 digits accuracy. We want to find a root $f(x) = x^2 - 2$. The Newton map is $T(x) = x - (x^2 - 2)/(2x)$. Lets start with $x = 1$.

$$\begin{aligned} T(1) &= 1 - (1 - 2)/2 = 3/2 \\ T(3/2) &= 3/2 - ((3/2)^2 - 2)/3 = 17/12 \\ T(17/12) &= 577/408 \\ T(577/408) &= 665857/470832. \end{aligned}$$

This is already $1.6 \cdot 10^{-12}$ close to the real root! 12 digits, by hand!

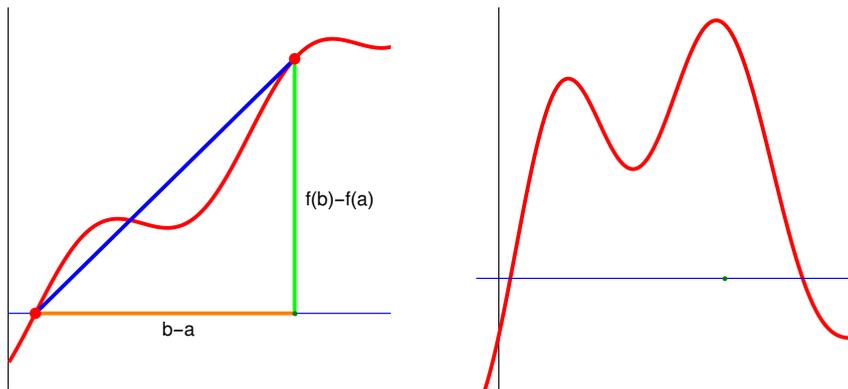
Example: To find the cube root of 10 we have to find a root of $f(x) = x^3 - 10$. The Newton map is $T(x) = x - (x^3 - 10)/(3x^2)$. If we start with $x = 2$, we get the following steps 2, 13/6, 3277/1521, 105569067476/49000820427. After three steps we have a result which is already $2.2 \cdot 10^{-9}$ close to the root.

Example: Verify that the Newton map $T(x)$ in the case $f(x) = a(x - b)^n$ with $n > 0$ has the property that for the root $x = b$ is obtained.

14.5. Unlike the intermediate value theorem which applied for continuous functions, the mean value theorem involves derivatives. We assume therefore today that all functions are differentiable unless specified.

Mean value theorem: Any interval (a, b) contains a point x such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}.$$



Here are a few examples which illustrate the theorem:

Example: Verify with the mean value theorem that the function $f(x) = x^2 + 4 \sin(\pi x) + 5$ has a point where the derivative is 1.

Solution. Since $f(0) = 5$ and $f(1) = 6$ we see that $(f(1) - f(0))/(1 - 0) = 1$.

Example: A biker drives with velocity $f'(t)$ at position $f(t)$ at time t and at position a at time a . The value $f(b) - f(a)$ is the distance traveled. The fraction $[f(b) - f(a)]/(b - a)$ is the average speed. The theorem tells that there was a time when the bike had exactly the average speed.

14.6. Proof of the theorem: the function $h(x) = f(a) + c(x - a)$, where $c = (f(b) - f(a))/(b - a)$ also connects the beginning and end point. The function $g(x) = f(x) - h(x)$ has now the property that $g(a) = g(b)$. If we can show that for such a function, there exists x with $g'(x) = 0$, then we are done. By tilting the picture, we have reduced the statement to

Rolle's theorem: If $f(a) = f(b)$ then f has a critical point in (a, b) .

Proof: If it were not true, then either $f'(x) > 0$ everywhere implying $f(b) > f(a)$ or $f'(x) < 0$ implying $f(b) < f(a)$.

Example: Show that the function $f(x) = \sin(x) + x(\pi - x)$ has a critical point $[0, \pi]$.

Solution: The function is differentiable and nonnegative. It is zero at $0, \pi$. By Rolle's theorem, there is a critical point.

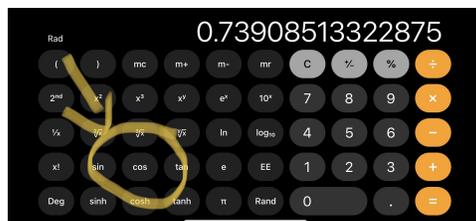
Example: Verify that the function $f(x) = 2x^3 + 3x^2 + 6x + 1$ has only one real root. **Solution:** There is at least one real root by the intermediate value theorem: $f(-1) = -4, f(1) = 12$. Assume there would be two roots. Then by Rolle's theorem there would be a value x where $g(x) = f'(x) = 6x^2 + 6x + 6 = 0$. But there is no root of g . [The graph of g minimum at $g'(x) = 6 + 12x = 0$ which is $1/2$ where $g(1/2) = 21/2 > 0$.]

Homework

Problem 14.1: Get the Newton map $T(x) = x - f(x)/f'(x)$ for:

- $f(x) = (x - 2)^2$
- $f(x) = e^{5x}$
- $f(x) = 2e^{-x^2}$
- $f(x) = \cot(x)$.

Problem 14.2: The function $f(x) = \cos(x) - x$ has a root between 0 and 1. We get closer to the root by doing two Newton steps starting with $x = 1$.



Compare with the root $x = 0.739085\dots$ obtained by punching "cos" again and again

Problem 14.3: We want to find the square root of 102. We have to solve $\sqrt{102} = x$ or $f(x) = x^2 - 102 = 0$. Perform two Newton steps starting at $x = 10$.

Problem 14.4: Find the Newton step $T(x) = x - f(x)/f'(x)$ in the case $f(x) = 1/x$. What happens if you apply the Newton steps starting with $x = 1$? Does the method converge?

Problem 14.5: We look at the function $f(x) = x^{10} + x^4 - 20x$ on the positive real line. Use the **mean value theorem** on $(1, 2)$ to assure there exists x , where $g(x) = f'(x) - [f(2) - f(1)] = f'(x) - 1018$. Now use one Newton step starting with 1.5 to find a solution to $g(x) = 0$

INTRODUCTION TO CALCULUS

MATH 1A

Unit 15: Review

MAJOR POINTS

f is **continuous** at a if there is $b = f(a)$ such that $\lim_{x \rightarrow a} f(x) = b$ for every a . The intermediate value theorem: $f(a) > 0, f(b) < 0$ implies f having a root in (a, b) .

$f'(x) = 0, f''(x) > 0$ then x is **local min**. $f'(x) = 0, f''(x) < 0$ then x is **local max**. For **global minima or maxima**, compare local extrema and boundary values.

If f changes sign we have a **root** $f = 0$, if f' changes sign, we have a **critical point** $f' = 0$ if f'' changes sign, we have an **inflection points**. A function is **even** if $f(-x) = f(x)$, and **odd** if $f(-x) = -f(x)$.

If $f' > 0$ then f is increasing, if $f' < 0$ it is decreasing. If $f''(x) > 0$ it is **concave up**, if $f''(x) < 0$ it is **concave down**. If $f'(x) = 0$ then f has a horizontal tangent.

Hospital's theorem applies for $0/0$ or ∞/∞ situations. In that case, $\lim_{x \rightarrow p} f(x)/g(x)$, where $f(p) = g(p) = 0$ or $f(p) = g(p) = \infty$ with $g'(p) \neq 0$ are given by $f'(p)/g'(p)$.

With $Df(x) = (f(x+h) - f(x))/h$ and $S(x) = h(f(0) + f(2h) + \dots + f((k-1)h))$ we have a **preliminary fundamental theorem of calculus** $SDf(kh) = f(kh) - f(0)$ and $DS(f(kh)) = f(kh)$.

Roots of $f(x)$ with $f(a) < 0, f(b) > 0$ can be obtained by the dissection method by applying the **Newton map** $T(x) = x - f(x)/f'(x)$ again and again.

Algebra reminders

Healing: $(a+b)(a-b) = a^2 - b^2$ or $1 + a + a^2 + a^3 + a^4 = (a^5 - 1)/(a - 1)$
Denominator: $1/a + 1/b = (a+b)/(ab)$
Exponential: $(e^a)^b = e^{ab}$, $e^a e^b = e^{a+b}$, $a^b = e^{b \log(a)}$
Logarithm: $\log(ab) = \log(a) + \log(b)$. $\log(a^b) = b \log(a)$
Trig functions: $\cos^2(x) + \sin^2(x) = 1$, $\sin(2x) = 2 \sin(x) \cos(x)$, $\cos(2x) = \cos^2(x) - \sin^2(x)$
Square roots: $a^{1/2} = \sqrt{a}$, $a^{-1/2} = 1/\sqrt{a}$

Important functions

Polynomials	$x^3 + 2x^2 + 3x + 1$	Exponential	$5e^{3x}$
Rational functions	$(x + 1)/(x^3 + 2x + 1)$	Logarithm	$\log(3x)$
Trig functions	$2 \cos(3x)$	Inverse trig functions	$\arctan(x)$

Important derivatives

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
$f(x) = x^n$	nx^{n-1}	$f(x) = \sin(ax)$	$a \cos(ax)$
$f(x) = e^{ax}$	ae^{ax}	$f(x) = \tan(x)$	$1/\cos^2(x)$
$f(x) = \cos(ax)$	$-a \sin(ax)$	$f(x) = \log(x)$	$1/x$
$f(x) = \arctan(x)$	$1/(1 + x^2)$	$f(x) = \sqrt{x}$	$1/(2\sqrt{x})$

Differentiation rules

Addition rule	$(f + g)' = f' + g'$.	Quotient rule	$(f/g)' = (f'g - fg')/g^2$.
Scaling rule	$(cf)' = cf'$.	Chain rule	$(f(g(x)))' = f'(g(x))g'(x)$.
Product rule	$(fg)' = f'g + fg'$.	Easy rule	simplify before deriving

Extremal problems

To maximize or minimize f on an interval $[a, b]$, find all critical points inside the interval, evaluate f on the boundary $f(a), f(b)$ and then compare the values to find the global maximum. Do not make the second derivative test at the boundary.

Limit examples

$\lim_{x \rightarrow 0} \sin(x)/x$	l'Hopital 0/0	$\lim_{x \rightarrow 1} (x^2 - 1)/(x - 1)$	heal
$\lim_{x \rightarrow 0} (1 - \cos(x))/x^2$	l'Hopital 0/0 twice	$\lim_{x \rightarrow \infty} \exp(x)/(1 + \exp(x))$	l'Hopital
$\lim_{x \rightarrow 0} (1/x)/\log(x)$	l'Hopital ∞/∞	$\lim_{x \rightarrow 0} (x + 1)/(x + 5)$	no work necessary

Important things

Summation and rate of change are at the heart of calculus.
 The 3 major types of discontinuities are jump, oscillation, infinity.
 Dissection and Newton methods are algorithms to find roots.
 The fundamental theorem of trigonometry is $\lim_{x \rightarrow 0} \sin(x)/x = 1$.
 The derivative is the limit $Df(x) = [f(x + h) - f(x)]/h$ as $h \rightarrow 0$.
 The rule $D(1 + h)^{x/h} = (1 + h)^{x/h}$ leads to $\exp'(x) = \exp(x)$.
 If you forget a derivative like of $\arctan(x)$, use the chain rule.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 16: Catastrophes

LECTURE

16.1. In this lecture, we are interested how minima and maxima change when a parameter changes. Nature, economies, processes favor extreme points. If optimal values change smoothly with parameters, how come that the outcome is often not smooth? What is the reason that an economic change can go so fast, once a tipping point is reached? One can explain this with mathematical models. A simple example explains the general principle:

If a local minimum disappears to some change of external parameter, the system settles in a new stable equilibrium. The new equilibrium can be far away from the original one.

16.2. To see this, let us look at the following optimization problem

Example: Find all the extrema of the function

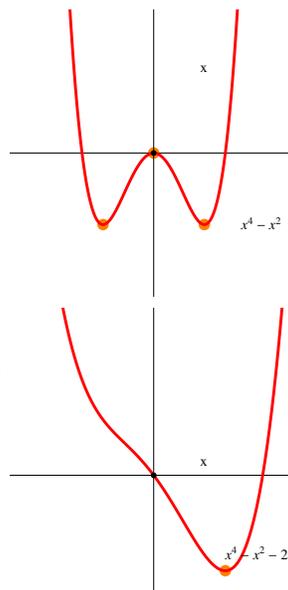
$$f(x) = x^4 - x^2$$

Solution: $f'(x) = 4x^3 - 2x$ is zero for $x = 0, 1/\sqrt{2}, -1/\sqrt{2}$. The second derivative is $12x^2 - 2$. It is negative for $x = 0$ and positive at the other two points. We have two local minima and one local maximum.

Example: Now find all the extrema of the function

$$f(x) = x^4 - x^2 - 2x$$

There is only one critical point. It is $x = 1$.

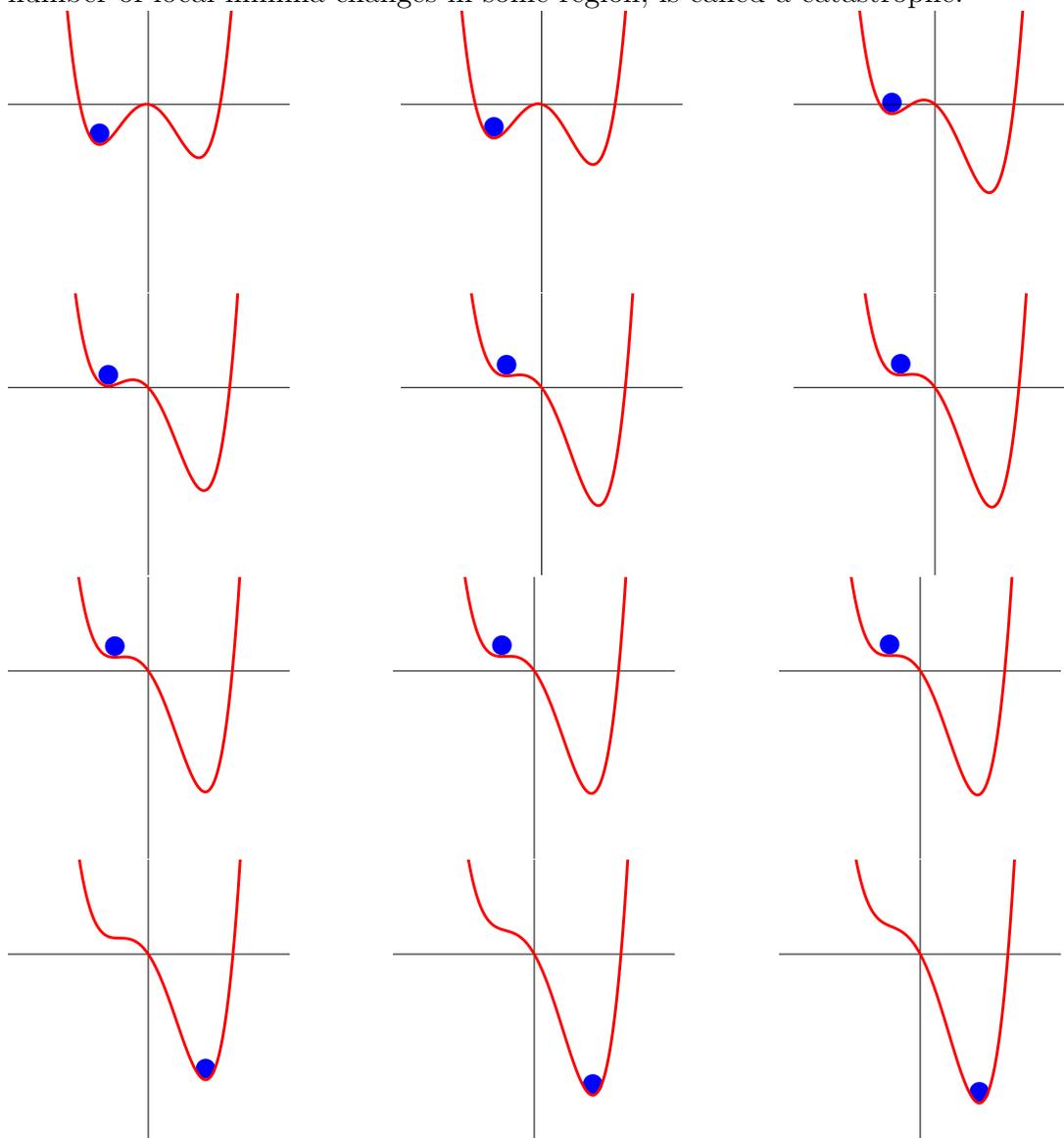


16.3. When the first graph is morphed to the second example, the local minimum to the left has disappeared. Assume the function f measures the prosperity of some kind and c is a **parameter**. We look at the position of the first critical point of the function. Catastrophe theorists look at the following **assumption**:

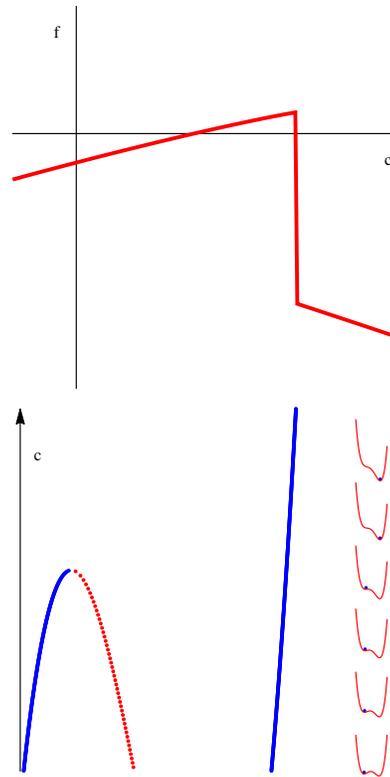
Definition: A **stable equilibrium** is a local minimum of the function. Assume the system depends on a parameter, then the minimum depends on this parameter. It remains a stable equilibrium until it disappears. If that happens, the system settles in a neighboring stable equilibrium.

Definition: A parameter value c_0 at which somewhere a stable minimum disappears, is called a **catastrophe**. In other words, if for $c < c_0$ a different collection of local minima exist than for $c > c_0$, then the parameter value c_0 is called a **catastrophe**.

16.4. In order to visualize a catastrophe, we draw the graphs of the function $f_c(x)$ for various parameters c and look at the local minima. At a parameter value, where the number of local minima changes in some region, is called a catastrophe.



You see that in this particular case, the catastrophe has happened between the 9th and 10th picture. Here is the position of the equilibrium point in dependence of c .



16.5. A **bifurcation diagram** displays the equilibrium points as they change in dependence of the parameter c . The vertical axes is the parameter c , the horizontal axes is x . At the bottom for $c = 0$, there are three equilibrium points, two local minima and one local maximum. At the top for $c = 1$ we have only one local minimum. Here is an important principle:

Catastrophes often lead to a strict and abrupt decrease of the minimal critical value. It is not possible to reverse the process in general.

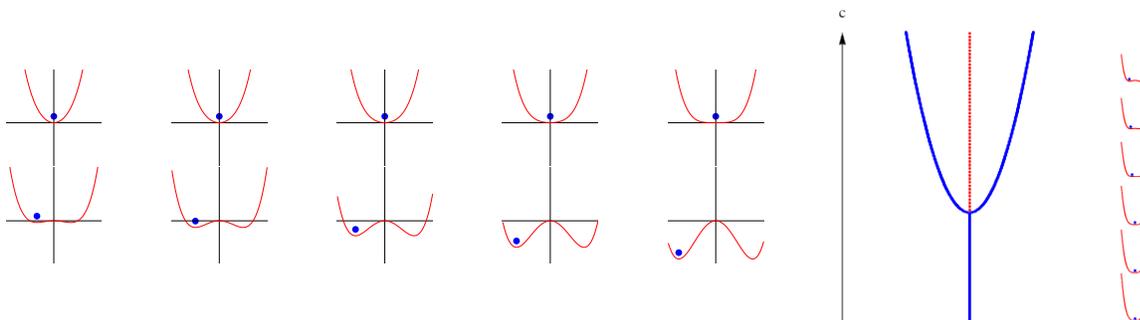
If we look at the above “movie” of graphs and run it backwards and use the same principle, we do not end up at the position we started with. The new equilibrium remains the equilibrium nearby.

Catastrophes are in general **irreversible**.

16.6. We know this from experience: it is easy to screw up a relationship, get sick, have a ligament torn or lose trust. Building up a relationship, getting healthy or gaining trust usually happens continuously and slowly. Ruining the economy of a country or a company or losing a good reputation of a brand can be quick. It takes time to regain it.

Local minima can change discontinuously, when a parameter is changed. This can happen with perfectly smooth functions and smooth parameter changes.

Example: We look at the example $f(x) = x^4 - cx^2$ with $-1 \leq c \leq 1$ in class.



Homework

In this homework, we study a catastrophe for the function

$$f(x) = x^6 - x^4 + cx^2,$$

where c is a parameter between 0 and 1.

Problem 16.1: a) Find all the critical points in the case $c = 0$ and analyze their stability. b) Find all the critical points in the case $c = 1$ and analyze their stability.

Problem 16.2: Plot the graph of the function $f(x)$ for 10 values of c between 0 and 1. You can use a graphing calculator or Wolfram alpha. Mathematica example code is below.

Problem 16.3: If you change from $c = -0.3$ to 0.6, pinpoint the value for the catastrophe and show a rough plot of $c \rightarrow f(x_c)$, the value at the first local minimum x_c in dependence of c . The text above provides this graph for an other function. It is the graph with a discontinuity.

Problem 16.4: If you change back from $c = 0.6$ to -0.3 pinpoint the value for the catastrophe. It will be different from the one in the previous question.

Problem 16.5: Sketch the bifurcation diagram. That is, if $x_k(c)$ is the k 'th equilibrium point, then draw the union of all graphs of $x_k(c)$ as a function of c (the c -axes pointing upwards). As in the two example provided, draw the local maximum with dotted lines.

```
Manipulate [ Plot [ x^6 - x^4 + c x^2, {x, -1, 1}], {c, 0, 1} ]
```

INTRODUCTION TO CALCULUS

MATH 1A

Unit 17: Riemann Integral

LECTURE

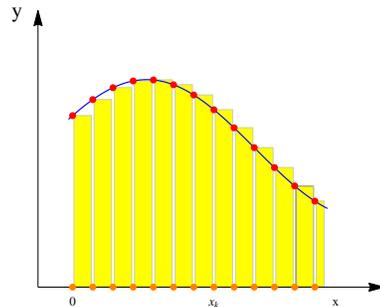
17.1. In this lecture, we define the definite integral $\int_0^x f(t) dt$ if f is a differentiable function. We then compute it for some basic functions. We have previously defined the **Riemann sums**

$$Sf(x) = h[f(0) + f(h) + f(2h) + \cdots + f(kh)] ,$$

where k is the largest integer such that $kh < x$. Lets write S_n if we want to stress that the parameter $h = 1/n$ was used in the sum. We define the **Riemann integral** as the limit of these sums $S_n f$, when the **mesh size** $h = 1/n$ goes to zero.

Definition: Define

$$\int_0^x f(t) dt = \lim_{n \rightarrow \infty} S_n f(x) .$$



For any differentiable function, the limit exists.

Proof: Assume first $f \geq 0$ on $[0, x]$. Let M be such that $f \leq M$ and $f' \leq M$ on $[0, x]$. The Riemann sum $S_n f(x)$ is the total area of K rectangles. Let $Sf(x)$ denote the area under the curve. If M is the maximal slope of f on $[0, x]$, then on each interval $[j/n, (j+1)/n]$, we have $|f(x) - f(j/n)| \leq M/n$ so that the area error is smaller than M/n^2 . As there are n such errors the error is smaller than $M(M/n^2) = M/n$. An additional rectangle above $[Kh, x]$ of area $\leq M/n$ is an upper bound on the discrepancy at the right boundary. If we add all the $k \leq xn$ “roof area errors” and the “side area” up, we get so

$$Sf(x) - S_n f(x) \leq \frac{kM}{n^2} + \frac{M}{n} \leq \frac{xnM}{n^2} + \frac{M}{n} = \frac{xM + M}{n} .$$

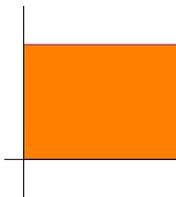
This converges to 0 for $n \rightarrow \infty$. The limit is therefore the area $Sf(x)$. For a general, not necessarily non-negative function, we write $f = g - h$, where g, h are non-negative and have $\int_0^x f(x) dx = \int_0^x g(x) dx - \int_0^x h(x) dx$.

For non-negative f , the value $\int_0^x f(x) dx$ is the **area between the x-axis and the graph** of f . For general f , it is a **signed area**, the difference between two areas.

17.2. Remark: the Riemann integral is defined here as the limit $h \sum_{x_k=kh \in [0,x]} f(x_k)$. It converges to the area under the curve for all **continuous** functions but since we work with differentiable functions in calculus we restricted to that. Not **all** bounded functions can be integrated naturally like this. There are discontinuous functions like the **salt and pepper function** which is defined to be $f(x) = 1$ if x is rational and zero else. Now $Sf(x) = 1$ for rational h and $Sf(x) = 0$ if h is irrational. Therefore, an other integral, the **Lebesgue integral** is used too: it can be defined as the limit $\frac{1}{n} \sum_{k=1}^n f(x_k)$ where x_k are **random points** in $[0, x]$. This **Monte-Carlo integral** definition of the Lebesgue integral gives the integral 0 for the salt and pepper function because rational numbers have zero probability.

17.3. Remark: The Riemann integral can be defined for partitions $x_0 < x_1 < \dots < x_n$ of points of the interval $[0, x]$ such that the maximal distance $(x_{k+1} - x_k)$ between neighboring x_j goes to zero. The Riemann sum is then $S_n f = \sum_k f(y_k)(x_{k+1} - x_k)$, where y_k is arbitrarily chosen inside the interval (x_k, x_{k+1}) . For continuous functions, the limiting result is the same the $Sf(x)$ sum done here. There are numerical reasons to allow more general partitions because it allows to adapt the mesh size: use more points where the function is complicated.

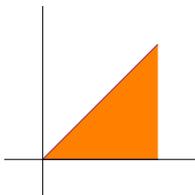
Example: If $f(x) = c$ is constant, then $\int_0^x f(t) dt = cx$. We can see also that $cnx/n \leq S_n f(x) \leq c(n+1)x/n$.



Example: Let $f(x) = cx$. The area is half of a rectangle of width x and height cx so that the area is $cx^2/2$. Adding up the Riemann sum is more difficult. Let k be the largest integer smaller than $xn = x/h$. Then

$$S_n f(x) = \frac{1}{n} \sum_{j=1}^k \frac{cj}{n} = \frac{ck(k+1)/2}{n^2}.$$

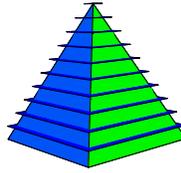
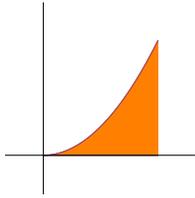
Taking the limit $n \rightarrow \infty$ and using that $k/n \rightarrow x$ shows that $\int_0^x f(t) dt = cx^2/2$.



Example: Let $f(x) = x^2$. In this case, we can not see the numerical value of the area geometrically. But since we have computed $S[x^2]$ in the first lecture of this course and seen that it is $[x^3]/3$ and since we have defined $S_h f(x) \rightarrow \int_0^x f(t) dt$ for $h \rightarrow 0$ and $[x^k] \rightarrow x^k$ for $h \rightarrow 0$, we know that

$$\int_0^x t^2 dt = \frac{x^3}{3}.$$

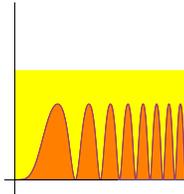
This example actually computes the **volume of a pyramid** which has at distance t from the top an area t^2 cross section. Think about $t^2 dt$ as a slice of the pyramid of area t^2 and height dt . Adding up the volumes of all these slices gives the volume.



Linearity of the integral (see homework) $\int_0^x f(t) + g(t) dt = \int_0^x f(t) dt + \int_0^x g(t) dt$
and $\int_0^x \lambda f(t) dt = \lambda \int_0^x f(t) dt$.

Upper bound: If $0 \leq f(x) \leq M$ for all x , then $\int_0^x f(t) dt \leq Mx$.

Example: $\int_0^x \sin^2(\sin(\sin(t))) / x dt \leq x$. **Solution.** The function $f(t)$ inside the interval is nonnegative and smaller or equal to 1. The graph of f is therefore contained in a rectangle of width x and height 1.



We see that if two functions are close then their difference is a function which is included in a small rectangle and therefore has a small integral:

If f and g satisfy $|f(x) - g(x)| \leq c$, then

$$\int_0^x |f(x) - g(x)| dx \leq cx.$$

We know identities like $S_n[x]_h^n = \frac{[x]_h^{n+1}}{n+1}$ and $S_n \exp_h(x) = \exp_h(x)$ already. Since $[x]_h^k - [x]^k \rightarrow 0$ we have $S_n[x]_h^k - S_n[x]^k \rightarrow 0$ and from $S_n[x]_h^k = [x]_h^{k+1} / (k+1)$. The other equalities are the same since $\exp_h(x) = \exp(x) \rightarrow 0$. This gives us:

$$\int_0^x t^n dt = \frac{x^{n+1}}{n+1}$$

$$\int_0^x e^t dt = e^x - 1$$

$$\int_0^x \cos(t) dt = \sin(x)$$

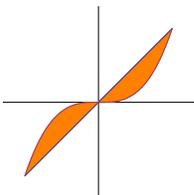
$$\int_0^x \sin(t) dt = 1 - \cos(x)$$

Homework

In the following homework you can use that $\int_a^b f(x) dx = F(b) - F(a)$ if F is a function which satisfies $F'(x) = f(x)$. We have already verified it for sums.

- Problem 17.1:** a) What is the integral $\int_1^2 x^8 dx$?
 b) Find the integral $\int_0^1 8t^7 + e^t dt$.
 c) Calculate $\int_{-1}^1 \frac{1}{1+x^2} dx$.
 d) Find $\int_0^{\pi/2} \sin^2(t) dt$. e) Find $\int_0^{\pi/2} \sin^4(t) dt$.

Problem 17.2: The region enclosed by the graph of x and the graph of x^3 has a propeller type shape as seen in the picture. Find its (positive) area.



Problem 17.3: Make a geometric picture for each of the following statements (which are rules for integration):

- $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$.
- $\int_a^b f(x) dx - \int_a^b g(x) dx = \int_a^b f(x) - g(x) dx$.
- $\int_a^b \lambda f(x) dx = \lambda \int_a^b f(x) dx$.

Problem 17.4: Here are some more challenging integrals. Maybe you have to guess:

- a) $\int_0^1 (3/2)\sqrt{1+x} dx$
 b) $\int_0^{\sqrt{\log(2)}} 16xe^{-x^2} dx$
 c) $\int_0^\pi \sin^4(x) dx$
 d) $\int_1^e 5 \log(x)/x dx$ For c), use double angle formulas, twice.

Problem 17.5: In this problem, it is crucial that you plot the function first. Split the integral up into parts.

- a) Find $\int_0^3 |x-1| dx$. Distinguish cases.
 b) Find $\int_0^3 f(x) dx$ for $f(x) = |x - |x-1||$. Also here, distinguish cases

INTRODUCTION TO CALCULUS

MATH 1A

Unit 18: Fundamental theorem

LECTURE

18.1. The **fundamental theorem of calculus** for differentiable functions allows us in general to compute integrals nicely. You have already made use of this theorem in the homework for today. Earlier in the course, we saw that $Sf(x) = h(f(0) + \dots + f(kh))$ and $Df(x) = (f(x+h) - f(x))/h$ we have $SDf = f(x) - f(0)$ and $DSf(x) = f(x)$ if $x = nh$. This now becomes the **fundamental theorem**. It assumes f' to be continuous.

$$\int_0^x f'(t) dt = f(x) - f(0) \text{ and } \frac{d}{dx} \int_0^x f(t) dt = f(x)$$

Proof. Using notation of Euler, we write $A \sim B$. We say "A and B are close" and mean that $A - B \rightarrow 0$ for $h \rightarrow 0$.¹ From $DSf(x) = f(x)$ for $x = kh$ we have $DSf(x) \sim f(x)$ for $kh < x < (k+1)h$ because f is continuous. We also know $\int_0^x Df(t) dt \sim \int_0^x f'(t) dt$ because $Df(t) \sim f'(t)$ uniformly for all $0 \leq t \leq x$ by the definition of the derivative and the assumption that f' is continuous and using Bolzano on the bounded interval. We also know $SDf(x) = f(x) - f(0)$ for $x = kh$. By definition of the Riemann integral, $Sf(x) \sim \int_0^x f(t) dt$ and so $SDf(x) \sim \int_0^x Df(t) dt$.

$$f(x) - f(0) \sim SDf(x) \sim \int_0^x Df(t) dt \sim \int_0^x f'(t) dt$$

as well as

$$f(x) \sim DSf(x) \sim D \int_0^x f(t) dt \sim \frac{d}{dx} \int_0^x f(t) dt .$$

Example: $\int_0^5 x^7 dx = \frac{x^8}{8} \Big|_0^5 = \frac{5^8}{8}$. You can always leave such expressions as your final result. It is even more elegant than the actual number 390625/8.

Example: $\int_0^{\pi/2} \cos(x) dx = \sin(x) \Big|_0^{\pi/2} = 1$.

Example: Find $\int_0^\pi \sin(x) dx$. **Solution:** The answer is 2.

Example: For $\int_0^2 \cos(t+1) dt = \sin(x+1) \Big|_0^2 = \sin(2) - \sin(1)$, the additional term +1 does not make matter as when using the chain rule, it goes away..

Example: $\int_{\pi/6}^{\pi/4} \cot(x) dx$. This is an example where the anti-derivative is difficult to spot. It becomes only easy when knowing where to look: the function $\log(\sin(x))$ has

¹Bolzano or Weierstrass would write $A \sim B$ as $\forall \epsilon > 0, \exists \delta > 0, |h| < \delta \Rightarrow |A - B| < \epsilon$. Parse this!

the derivative $\cos(x)/\sin(x)$. So, we know the answer is $\log(\sin(x))|_{\pi/6}^{\pi/4} = \log(\sin(\pi/4)) - \log(\sin(\pi/6)) = \log(1/\sqrt{2}) - \log(1/2) = -\log(2)/2 + \log(2) = \log(2)/2$.

Let us look at two for now more challenging cases:

Example: The example $\int_2^3 2/(t^2 - 1) dt$ is challenging for now. We need a hint and write $2/(x^2 - 1) = 1/(x - 1) - 1/(x + 1)$. The function $F(x) = \log|x - 1| - \log|x + 1|$ has therefore $f(x) = 2/(x^2 - 1)$ as a derivative. The answer is $\int_2^3 2/(t^2 - 1) dt = F(3) - F(2) = \log(2) - \log(4) - \log(1) + \log(3) = \log(3) - \log(2) = \log(3/2)$.

Example: $\int_0^x \cos(\sin(x)) \cos(x) dx = \sin(\sin(x))$ because the derivative of $\sin(\sin(x))$ is $\cos(\sin(x)) \cos(x)$. The function $\sin(\sin(x))$ is an **anti-derivative** of f . If we differentiate this function, we get $\cos(\sin(x)) \cos(x)$. Also this can be hard to spot for now. We will learn how to do this

Here is an important notation, which we have used in the example and which might at first look silly. But it is a handy intermediate step when doing the computation.

Definition: $F|_a^b = F(b) - F(a)$.

We give reformulations of the fundamental theorem in ways in which it is mostly used: If f is the derivative of a function F then

$$\int_a^b f(x) dx = F(x)|_a^b = F(b) - F(a) .$$

In some textbooks, this is called the "second fundamental theorem" or the "evaluation part" of the fundamental theorem of calculus. The statement $\frac{d}{dx} \int_0^x f(t) dt = f(x)$ is the "antiderivative part" of the fundamental theorem. They obviously belong together and are two different sides of the same coin.

Here is a version of the fundamental theorem, where the boundaries are functions of x . Given functions g, h and if F is a function such that $F' = f$, then

$$\int_{h(x)}^{g(x)} f(t) dt = F(g(x)) - F(h(x)) .$$

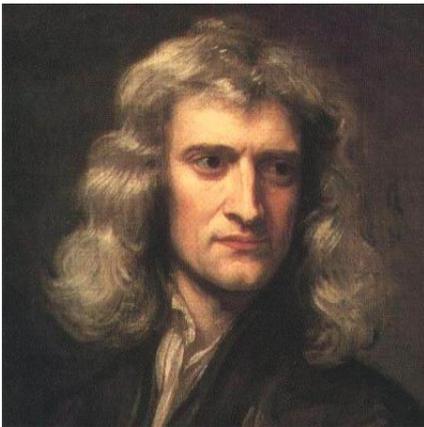
Example: $\int_{x^4}^{x^2} \cos(t) dt = \sin(x^2) - \sin(x^4)$.

The function F is called an **anti-derivative**. It is not unique but the above formula does always give the right result. Lets make a list You should have as many **anti-derivatives** "hard wired" in your brain. It really helps. Here are the core functions you should know.

INTRODUCTION TO CALCULUS

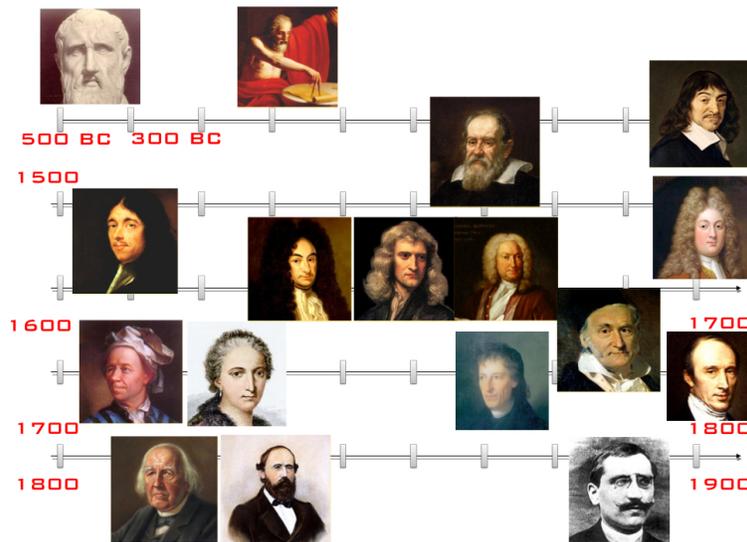
function	anti derivative
x^n	$\frac{x^{n+1}}{n+1}$
\sqrt{x}	$\frac{x^{3/2}}{3/2}$
e^{ax}	$\frac{e^{ax}}{a}$
$\cos(ax)$	$\frac{\sin(ax)}{a}$
$\sin(ax)$	$-\frac{\cos(ax)}{a}$
$\frac{1}{x}$	$\log(x)$
$\frac{1}{1+x^2}$	$\arctan(x)$
$\log(x)$	$x \log(x) - x$

Make your own table!



Newton (“Sir Slightly Annoyed”) and Leibniz (“Mr Sour Face”)

Meet **Isaac Newton** and **Gottfried Leibniz**. They have discovered the fundamental theorem of calculus. You can see from the expression of their faces, they are not pleased that Oliver has added other calculus pioneers. The sour faces might also have to do with the fact that they have to live already for 9 years on the same handout with **Austin Powers** and **Doctor Evil**! But that is ok. Celebrities can afford to suffer.



Homework

Problem 18.1: Find a function F such that $F' = f$, then integrate

a) $\int_0^2 4x^3 + 10x \, dx$.

b) $\int_0^1 (x+1)^3 \, dx$.

Problem 18.2: Find a function F such that $F' = f$, then integrate:

a) $\int_2^3 5/(x-1) \, dx$,

b) $\int_0^{\sqrt{\pi}} \sin(x^2)x \, dx$

Problem 18.3: Evaluate the following integrals:

a) $\int_1^2 2^x \, dx$

b) $\int_0^1 \frac{1}{1+x^2} \, dx$,

Problem 18.4: a) Compute $F(x) = \int_0^{x^3} \sin(t) \, dt$, then find $F'(x)$.

b) Compute $G(x) = \int_{\sin(x)}^{\cos(x)} \exp(t) \, dt$ then find $G'(x)$

Problem 18.5: a) **A clever integral:** Evaluate the following integral (just by being clever no algebra, and no work is needed):

$$\int_{-\pi}^{\pi} \sin(\sin(\sin(\sin(\sin(x)))))) \, dx .$$

b) **An evil integral:** Evaluate $\int_e^{e^e} \frac{1}{\log(x)x} \, dx$.

Hint: Figure out a function $F(x)$ which satisfies $F'(x) = 1/(\log(x)x)$. Don't hesitate to ask MiniMe (Oliver).



Clever



and evil

INTRODUCTION TO CALCULUS

MATH 1A

Unit 19: Anti-derivatives

LECTURE

19.1. The definite integral $\int_a^b f(t) dt$ is a **signed area under the curve**. We say “signed” because the area of the region below the curve is counted negatively. There is something else to mention: ¹

Definition: For every C , the function $F(x) = \int_0^x f(t) dt + C$ is called an **anti-derivative** of g . The constant C is arbitrary and not fixed.

19.2. The fundamental theorem of calculus assured us that

The anti derivative gives us from a function f a function F which has the property that $F' = f$. Two different anti derivatives F differ only by a constant.

19.3. Finding the anti-derivative of a function is in general harder than finding the derivative. We will learn some techniques but it is in general not possible to give anti derivatives for a function, if it looks simple.

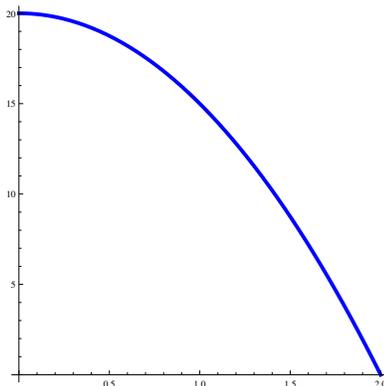
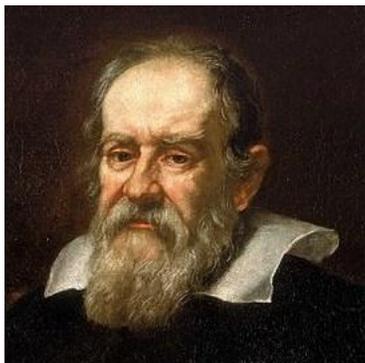
Example: Find the anti-derivative of $f(x) = \sin(4x) + 20x^3 + 1/x$. **Solution:** We can take the anti-derivative of each term separately. It is $F(x) = -\cos(4x)/4 + 4x^4 + \log(x) + C$.

Example: Find the anti derivative of $f(x) = 1/\cos^2(x) + 1/(1-x)$. **Solution:** we can find the anti-derivatives of each term separately and add them up. The result is $F(x) = \tan(x) + \log|1-x| + C$.

Example: **Galileo** measured **free fall**, a motion with constant acceleration. Assume $s(t)$ is the height of the ball at time t . Assume the ball has zero velocity initially and is located at height $s(0) = 20$. We know that the velocity is $v(t)$ is the derivative of $s(t)$ and the acceleration $a(t)$ is constant equal to -10 . So, $v(t) = -10t + C$ is the antiderivative of a . By looking at v at time $t = 0$ we see that $C = v(0)$ is the initial velocity and so zero. We know now $v(t) = -10t$. We need now to compute the anti derivative of $v(t)$. This is $s(t) = -10t^2/2 + C$. Comparing $t = 0$ shows $C = 20$. Now $s(t) = 20 - 5t^2$. The graph of s is a parabola. If we give the ball an additional

¹The definition of anti-derivative was merged also to unit 18 in 2020 due to move-out from campus.

horizontal velocity, such that time t is equal to x then $s(x) = 20 - 5x^2$ is the visible trajectory. We see that jumping from 20 meters leads to a fall which lasts 2 seconds.



Example: The **total cost** is the anti-derivative of the **marginal cost** of a good. Both the marginal cost as well as the total cost are a function of the quantity produced. For instance, suppose the total cost of making x shoes is 300 and the total cost of making $x + 4$ shoes is 360 for all x . The marginal cost is $60/4 = 15$ dollars. In general the marginal cost changes with the number of goods. There is additional cost needed to produce one more shoe if 300 shoes are produced. **Problem:** Assume the marginal cost of a book is $f(x) = 5 - x/100$ and that producing the first 10 books costs 1000 dollars. What is the total cost of producing 100 books? **Answer:** The anti derivative $5 - x/100$ of f is $F(x) = 5x - x^2/100 + C$ where C is a constant. By comparing $F(10) = 1000$ we get $50 - 100/100 + C = 1000$ and so $C = 951$. the result is $951 + 5 * 100 - 10'000/100 = 1351$. The average book prize has gone down from 100 to 13.51 dollars.

Example: The **total revenue** $F(x)$ is the anti-derivative of the **marginal revenue** $f(x)$. Also these functions depend on the quantity x produced. We have $F(x) = P(x)x$, where $P(x)$ is the prize. Then $f(x) = F'(x) = P'(x)x + P$. For a **perfect competitive market**, $P'(x) = 0$ so that the prize is equal to the marginal revenue.

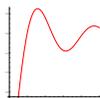
Definition: A function f is called **elementary**, if it can be constructed using addition, subtraction, multiplication, division, compositions from polynomials or roots. In other words, an elementary function is built up with functions like x^3 , $\sqrt{\cdot}$, \exp , \log , \sin , \cos , \tan and \arcsin , \arccos , \arctan .

Example: The function $f(x) = \sin(\sin(\pi + \sqrt{x} + x^2)) + \log(1 + \exp((x^6 + 1)/(x^2 + 1))) + (\arctan(e^x))^{1/3}$ is an elementary function.

Example: The anti derivative of the sinc function is called the **sine-integral**

$$Si(x) = \int_0^x \frac{\sin(t)}{t} dt .$$

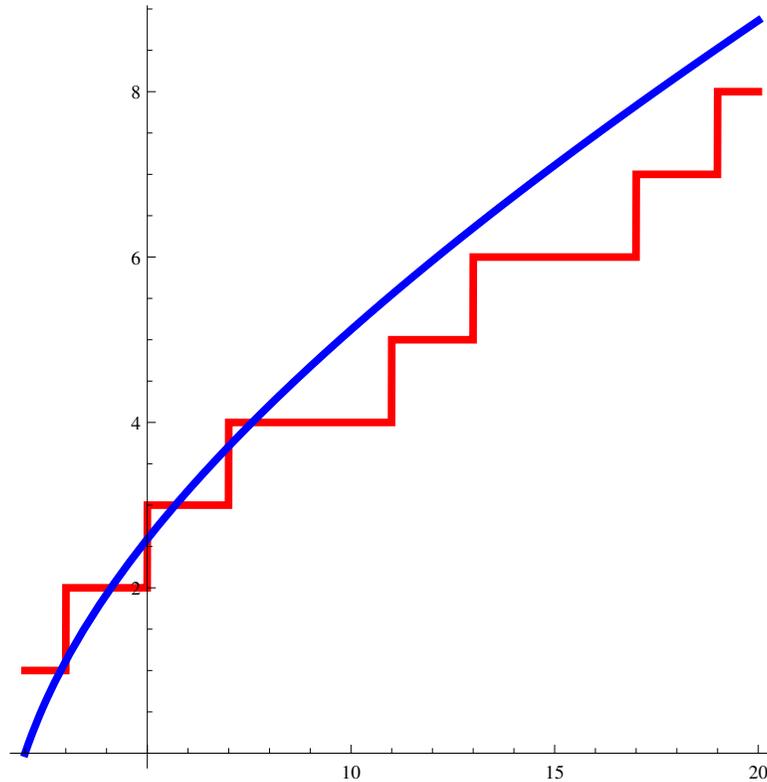
The function $Si(x)$ is not an elementary function.



Example: The **offset logarithmic integral** is defined as

$$\text{Li}(x) = \int_2^x \frac{dt}{\log(t)}$$

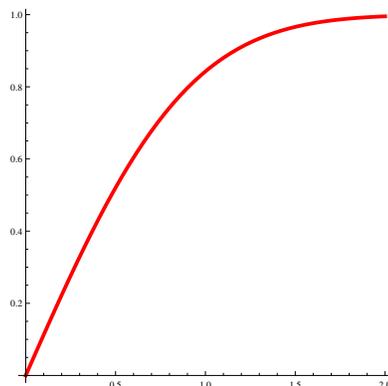
It is a specific anti-derivative. It is a good approximation of the number of prime numbers less than x . The graph below illustrates this. The second stair graph shows the number $\pi(x)$ of primes below x . For example, $\pi(10) = 4$ because 2, 3, 5, 7 are the only primes below it. The function Li is not an elementary function.



Example: The **error function**

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

is important in statistics. It is not an elementary function.



The Mathematica command "Integrate" uses about 500 pages of Mathematica code and 600 pages of C code.² Before software was doing this, tables of integrals were used. These were thousands of pages thick books contains some integrals, which computer algebra systems have trouble with.

There are other integrals we can do, but Mathematica can not do. Especially definite integrals like $\int_0^{2\pi} \sin(\sin(\sin(x))) dx$.

Numerical evaluation

What do we do when we have can not find the integral analytically? We can still compute it numerically. Here is an example: the function $f(x) = \sin(\sin(x))$ also does not have an elementary anti-derivative. But you could compute the integral $\int_0^x f(x) dx$ numerically with a computer algebra system like Mathematica:

```
NIntegrate [ Sin [ Sin [ x ] ] , { x , 0 , 10 } ]
```

One can approximate such a function also using trigonometric Polynomials and then integrate those. In the case, $\sin(\sin(x))$, the function $0.88\text{Sin}[x]+0.04\text{Sin}[3x]$ is already very close.

Pillow problems

Here are some integration riddles to ponder. They are just for fun. We will learn techniques to deal with them. They make also good pillow problems, problems to think about while falling asleep. Try it. Sometimes, you might know the answer in the morning. Maybe you can guess a function which has $f(x)$ as a derivative.

Problem 19.1: $f(x) = \cos(\log(x))/x$.

Problem 19.2: $f(x) = \frac{1}{x^4-1}$.

Problem 19.3: $f(x) = \cot^2(x)$.

Problem 19.4: $f(x) = \cos^4(x)$.

Problem 19.5: $f(x) = \frac{1}{x \log(x) \log(\log(x))}$.

²<http://reference.wolfram.com/legacy/v3/MainBook/A.9.5.html>

INTRODUCTION TO CALCULUS

MATH 1A

Unit 20: Area

LECTURE

20.1. If $f(x) \geq 0$, then $\int_a^b f(x) dx$ is the **area under the graph** of $f(x)$ and above the interval $[a, b]$ on the x axes. If the function is negative, then $\int_a^b f(x) dx$ is negative too and the integral is minus the area below the curve:

Therefore, $\int_a^b f(x) dx$ is the difference of the area above the graph minus the area below the graph.

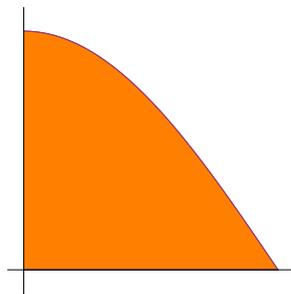
20.2. More generally we can also look at **areas sandwiched between two graphs f and g** .

The area of a region G enclosed by two graphs $f \leq g$ and bound by $a \leq x \leq b$ is

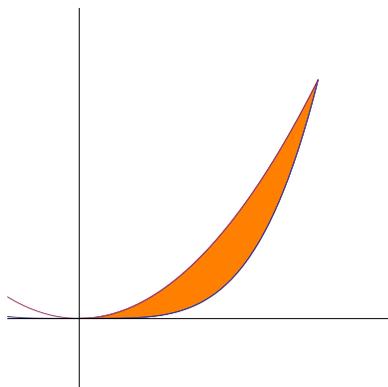
$$\int_a^b g(x) - f(x) dx$$

20.3. Make sure that if you have to compute such an integral that $g \geq f$ before giving it the interpretation of an area.

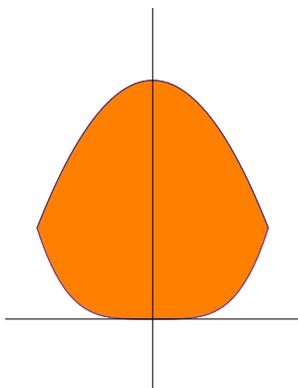
Example: Find the area of the region enclosed by the x -axes, the y -axes and the graph of the \cos function. **Solution:** $\int_0^{\pi/2} \cos(x) dx = 1$.



Example: Find the area of the region enclosed by the graphs $f(x) = x^2$ and $f(x) = x^4$.



Example: Find the area of the region enclosed by the graphs $f(x) = 1 - x^2$ and $g(x) = x^4$. **Solution:** The intersection points are $\pm(\sqrt{5} - 1)/2$ and called golden ratio. Now it is routine.

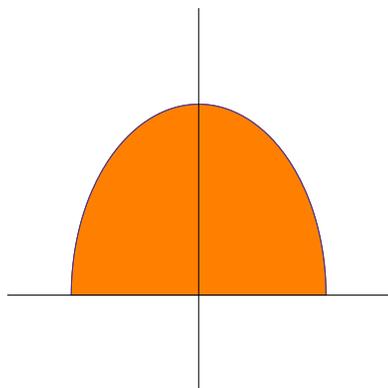


Example: Find the area of the region enclosed by a half circle of radius 1. **Solution:** The half circle is the graph of the function $f(x) = \sqrt{1 - x^2}$. The area under the graph is

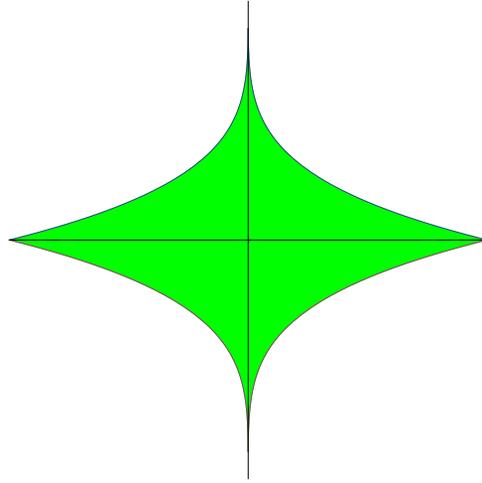
$$\int_{-1}^1 \sqrt{1 - x^2} dx .$$

Finding the anti-derivative is not so easy. We will find techniques to do so, for now we just are told to look at the derivative of $x\sqrt{1 - x^2} + \arcsin(x)$ and see what happens. With this “inspiration”, we find the anti derivative to be $(x\sqrt{1 - x^2} + \arcsin(x))/2$. The area is therefore

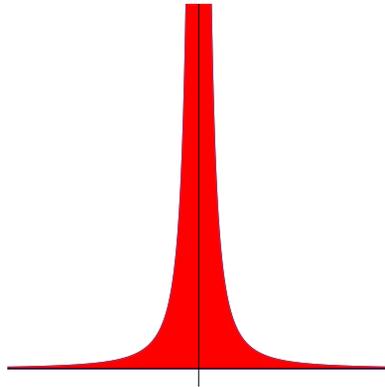
$$\frac{x\sqrt{1 - x^2} + \arcsin(x)}{2} \Big|_{-1}^1 = \frac{\pi}{2} .$$



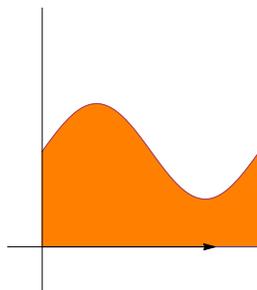
Example: Find the area of the region between the graphs of $f(x) = 1 - |x|^{1/4}$ and $g(x) = -1 + |x|^{1/4}$.



Example: Find the area under the curve of $f(x) = 1/x^2$ between -6 and 6 . Naive solution attempt. $\int_{-6}^6 x^{-2} dx = -x^{-1}|_{-6}^6 = -1/6 - 1/6 = -1/3$. There is something fishy with this computation because $f(x)$ is nonnegative so that the area should be positive. But we obtained a negative answer. What is going on?



Example: Find the area between the curves $x = 0$ and $x = 2 + \sin(y)$, $y = 2\pi$ and $y = 0$. **Solution:** We turn the picture by 90 degrees so that we compute the area under the curve $y = 0$, $y = 2 + \sin(x)$ and $x = 2\pi$ and $x = 0$.



Example: **The grass problem.** Find the area between the curves $|x|^{1/3}$ and $|x|^{1/2}$.

Solution. This example illustrates how important it is to have a picture. This is good advice for any "word problem" in mathematics.

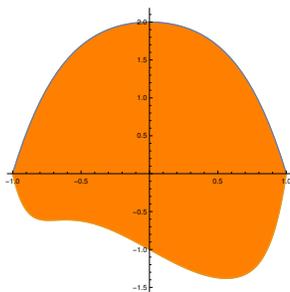
Use a picture of the situation while doing the computation.

Homework

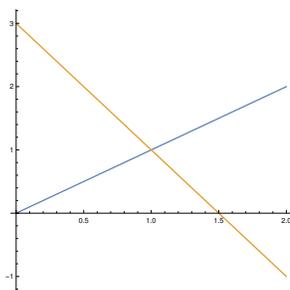
Problem 20.1: Find the area of the bounded region enclosed by the graphs $f(x) = 2x^5 - 24x$ and $g(x) = 4x^2$ for $x > 0$. It is a good idea to make a picture.

Problem 20.2: Find the area of the region enclosed by the curves $x = 0$, $x = \pi/2$, $y = 4 + \sin(5x)$, $y = \sin^2(2x)$.

Problem 20.3: Find the area of the region enclosed by the graphs $2 - x^4 - x^2$ and $x^{10} - 1 + x^3 - x$.



Problem 20.4: Find the area of the region enclosed by the three lines $y = x$, $y = 3 - 2x$ and $y = 0$.



Problem 20.5: Write down an integral which gives the area of the **area 51** region $x^2 + |y|^{51} \leq 1$ by writing the region as a sandwich between two graphs. Evaluate the integral numerically using Wolfram alpha, Mathematica or any other software.

INTRODUCTION TO CALCULUS

MATH 1A

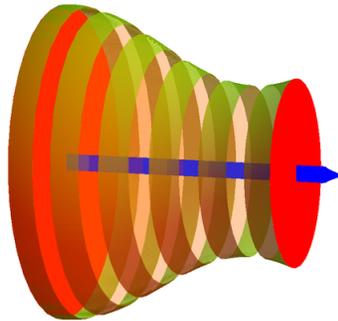
Unit 21: Volume

LECTURE

21.1. To compute the **volume of a solid**, we cut it into slices, where each slice is perpendicular to a given line x . If $A(x)$ is the **area of the slice** and the body is enclosed between a and b then

$$V = \int_a^b A(x) dx$$

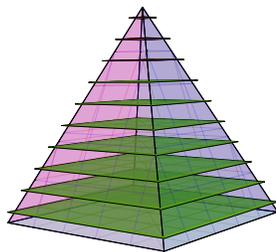
is the **volume**. Think of $A(x)dx$ as the volume of a slice. The integral adds up.



Example: Compute the volume of a pyramid with square base length 2 and height 2. **Solution:** we can assume the pyramid is built over the square $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. The cross section area at height h is $A(h) = (2 - h)^2$. Therefore,

$$V = \int_0^2 (2 - h)^2 dh = \frac{8}{3}.$$

This is base area 4 times height 2 divided by 3.



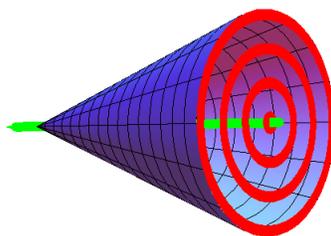
Definition: A **solid of revolution** is a surface enclosed by the surface obtained by rotating the graph of a function $f(x)$ around the x -axis.

The area of the cross section at x of a solid of revolution is $A(x) = \pi f(x)^2$. The volume of the solid is $\int_a^b \pi f(x)^2 dx$.

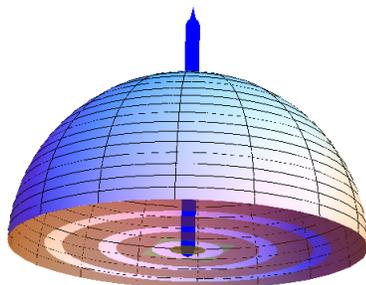
Example: Find the volume of a **round cone** of height 2 and where the circular base has the radius 1. **Solution.** This is a solid of revolution obtained by rotation the graph of $f(x) = x/2$ around the x axes. The area of a cross section is $\pi x^2/4$. Integrating this up from 0 to 2 gives

$$\int_0^2 \pi x^2/4 dx = \frac{x^3}{4 \cdot 3} \Big|_0^2 = \frac{2\pi}{3}.$$

This is the height 2 times the base area π divided by 3.

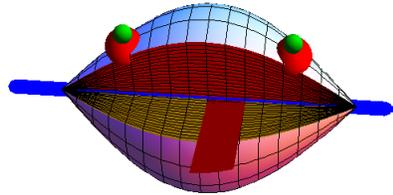


Example: Find the volume of a **half sphere** of radius 1. **Solution:** The area of the cross section at height h is $\pi(1 - h^2)$.



Example: If the function $f(x) = \sin(x)$ is rotated around the x axes, we get a **lemon**. But now we cut out a slice of $60 = \pi/3$ degrees as in the picture. Find the volume of the solid.

Solution: The area of a slice without the missing piece is $\pi \sin^2(x)$. The integral $\int_0^\pi \sin^2(x) dx$ is $\pi/2$ as derived in the lecture. Having cut out $1/6$ 'th the area is $(5/6)\pi \sin^2(x)$. The volume is $\int_0^\pi (5/6)\pi \sin^2(x) dx = (5/6)\pi^2/2$.

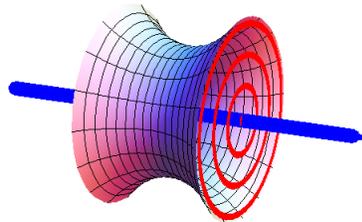


Homework

Problem 1: Find the volume of the **paraboloid** for which the radius at position x is $4 - x^2$ and x ranges from 0 to 2.

Problem 2: A **catenoid** is the surface obtained by rotating the graph of $f(x) = \cosh(x) = (\exp(x) + \exp(-x))/2$ around the x -axes. We have seen that the graph of f is the chain curve, the shape of a hanging chain. Find the volume of of the solid enclosed by the catenoid between $x = -2$ and $x = 2$.

Hint. You might want to check first the identity $\cosh(x)^2 = (1 + \cosh(2x))/2$ using the definition $\cosh(x) = (\exp(x) + \exp(-x))/2$.



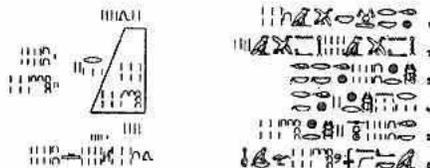
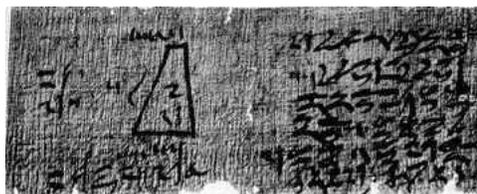
Problem 3: A **tomato** is given by $z^2 + x^2 + 4y^2 = 1$. If we slice perpendicular to the y axes, we get a circular slice $z^2 + x^2 \leq 1 - 4y^2$ of radius $\sqrt{1 - 4y^2}$. Find the area of this slice, then determine the volume of the tomato.

Problem 4: Archimedes was so proud of his formula for the volume of a sphere that he wanted the formula displayed on his tomb stone. To derive the formula, he wrote the volume of a half sphere of radius 1 as the difference between the volume of a cylinder of radius 1 and height 1 and the volume of a cone of base radius 1 and height 1. Relate the cross section area of the cylinder-cone complement with the cross section area of the sphere to recover his argument! If stuck, draw in the sand, soak in the bath tub or eat a tomato salad. No credit is given for screaming “Eureka”.

Problem 5: Volumes were among the first quantities, Mathematicians wanted to measure and compute. One problem on **Moscow Egypt papyrus** dating back to 1850 BC explains the general formula $h(a^2 + ab + b^2)/3$ for a **truncated pyramid** with base length a , roof length b and height h . Verify that if you slice such a **frustrum** at height x , the area is $A(x) = (a + (b - a)x/h)^2$. Now use this to compute the volume using calculus.

Here is the translated formulation from the papyrus: ¹ ²

Remark: ”You are given a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. You are to square this 4 result 16. You are to double 4 result 8. You are to square 2, result 4. You are to add the 16, the 8 and the 4, result 28. You are to take one-third of 6 result 2. You are to take 28 twice, result 56. See it is 56. You will find it right”.



OLIVER KNILL, KNILL@MATH.HARVARD.EDU, MATH 1A, HARVARD COLLEGE, SPRING 2020

¹Howard Eves, Great moments in mathematics, Volume 1, MAA, Dolciani Mathematical Expositions, 1980, page 10

²Image Source: http://www-history.mcs.st-and.ac.uk/HistTopics/Egyptian_papyri.html

INTRODUCTION TO CALCULUS

MATH 1A

Unit 22: Improper Integrals

LECTURE

22.1. In this lecture, we look at integrals on infinite intervals or integrals, where the function can get infinite at some point. These integrals are called **improper integrals**. The area under the curve can remain finite or become infinite.

Example: What is the integral

$$\int_1^{\infty} \frac{1}{x^2} dx ?$$

Since the anti-derivative is $-1/x$, we have

$$\left. \frac{-1}{x} \right|_1^{\infty} = -1/\infty + 1 = 1 .$$

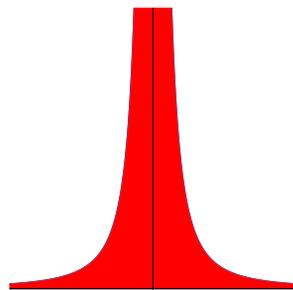
To justify this, compute the integral $\int_1^b 1/x^2 dx = 1 - 1/b$ and see that in the limit $b \rightarrow \infty$, the value 1 is achieved.

22.2. In a previous lecture, we have seen a shocking example similar to the following one:

Example:

$$\int_{-1}^1 \frac{1}{x^2} dx = \left. -\frac{1}{x} \right|_{-1}^1 = -1 - 1 = -2 .$$

This does not make any sense because the function is positive so that the integral should be a positive area. The problem is this time not at the boundary $-1, 1$. The sore point is $x = 0$ over which we have carelessly integrated over.



22.3. The next example illustrates the problem with the previous example better:

Example: The computation

$$\int_0^1 \frac{1}{x^2} dx = -\frac{1}{x} \Big|_0^1 = -1 + \infty .$$

indicates that the integral does not exist. We can justify by looking at integrals

$$\int_a^1 \frac{1}{x^2} dx = -\frac{1}{x} \Big|_a^1 = -1 + \frac{1}{a}$$

which are fine for every $a > 0$. But this does not converge for $a \rightarrow 0$. Now

$$\int_{-1}^{-a} \frac{1}{x^2} dx = -\frac{1}{x} \Big|_{-1}^{-a} = -1 + \frac{1}{a} .$$

If we add up both, we get $-2 + 2/a$. This value is positive for every $0 < a < 1$ but it does not disappear for $a \rightarrow 0$.

22.4. Do we always have a problem if the function goes to infinity at some point?

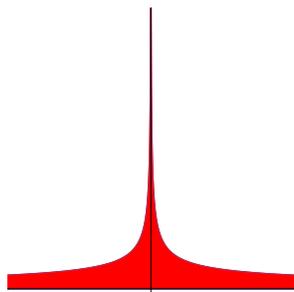
Example: Find the following integral

$$\int_0^1 \frac{1}{\sqrt{x}} dx .$$

Solution: Since the point $x = 0$ is problematic, we integrate from a to 1 with positive a and then take the limit $a \rightarrow 0$. Since $x^{-1/2}$ has the anti-derivative $x^{1/2}/(1/2) = 2\sqrt{x}$, we have

$$\int_a^1 \frac{1}{\sqrt{x}} dx = 2\sqrt{x} \Big|_a^1 = 2\sqrt{1} - 2\sqrt{a} = 2(1 - \sqrt{a}) .$$

There is no problem with taking the limit $a \rightarrow 0$. The answer is 2. Even so the region is infinite its area is finite. This is an interesting example. Imaging this to be a container for paint. We can fill the container with a finite amount of paint but the wall of the region has infinite length.



Example: Evaluate the integral $\int_0^1 1/\sqrt{1-x^2} dx$. **Solution:** The anti-derivative is $\arcsin(x)$. In this case, it is not the point $x = 0$ which produces the difficulty. It is the point $x = 1$. Take $a > 0$ and evaluate

$$\int_0^{1-a} \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) \Big|_0^{1-a} = \arcsin(1-a) - \arcsin(0) .$$

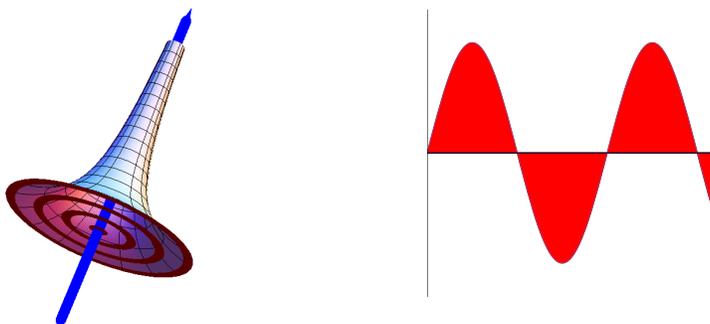
Now $\arcsin(1-a)$ has no problem at limit $a \rightarrow 0$. Since $\arcsin(1) = \pi/2$ exists. We get therefore the answer $\arcsin(1) = \pi/2$.

Example: Rotate the graph of $f(x) = 1/x$ around the x -axes and compute the volume of the solid between 1 and ∞ . The cross section area is π/x^2 . If we look at the integral from 1 to a fixed R , we get

$$\int_1^R \frac{\pi}{x^2} dx = -\frac{\pi}{x} \Big|_1^R = -\pi/R + \pi .$$

This converges for $R \rightarrow \infty$. The volume is π . This famous solid is called **Gabriel's trumpet**. This solid is so prominent because if you look at the surface area of the small slice, then it is larger than $dx2\pi/x$. The total surface area of the trumpet from 1 to R is therefore larger than $\int_1^R 2\pi/x dx = 2\pi(\log(R) - \log(1))$. which goes to infinity. We can **fill** the trumpet with a finite amount of paint but we can not **paint** its surface.

Example: Evaluate the integral $\int_0^\infty \sin(x) dx$. **Solution.** There is no problem at the boundary 0 nor at any other point. We have to investigate however, what happens at ∞ . Therefore, we look at the integral $\int_0^b \sin(x) dx = -\cos(x)|_0^b = 1 - \cos(b)$. We see that the limit $b \rightarrow \infty$ does not exist. The integral fluctuates between 0 and 2.



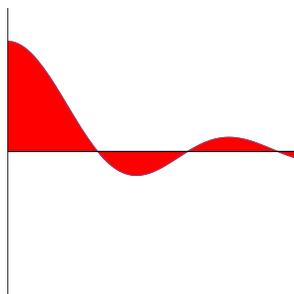
22.5. The next example leads to a topic in a follow-up course. It is not covered here, but could make you curious:

Example: What about the integral

$$I = \int_0^\infty \frac{\sin(x)}{x} dx ?$$

Solution. The anti derivative is the Sine integral $Si(x)$ so that we can write $\int_0^b \sin(x)/x dx = Si(b)$. It turns out that the limit $b \rightarrow \infty$ exists and is equal to $\pi/2$ but this is a topic for a second semester course like Math 1b. The integral can be written as an alternating series, which converges and there are many ways to compute it: ¹

¹Hardy, Mathematical Gazette, 5, 98-103, 1909.

**22.6.**

$\int_a^\infty f(x) dx$ is defined as $\lim_{t \rightarrow \infty} \int_a^t f(x) dx$ if the limit exists.
 $\int_0^b f(x) dx$ is defined as $\lim_{t \rightarrow 0} \int_t^b f(x) dx$ if the limit exists

Homework

Problem 22.1: Evaluate the integral $\int_0^1 x^{2/3} dx$.

Problem 22.2: For which $0 < p < \infty$ does the integral $\int_1^\infty 1/x^p dx$ exist? To investigate this, look at $\int_1^t 1/x^p dx$ and decide in which case the limit $t \rightarrow \infty$ exists.

Problem 22.3: Evaluate the improper integral $\int_{-1}^1 1/\sqrt{1-x^2} dx$. This example is related to the arcsin distribution in probability theory. Guess where this name comes from?

Problem 22.4: Evaluate the integral $\int_{-3}^4 (x^2)^{1/3} dx$. To make sure that the integral is fine, check separately whether \int_{-3}^0 and \int_0^4 work.

The integral $\int_{-2}^1 1/x dx$ does not exist. We can however take a positive $a > 0$ and look at

$$\int_{-2}^{-a} 1/x dx + \int_a^1 1/x dx = \log |a| - \log |-2| + (\log |1| - \log |a|) = \log(2).$$

If the limit exists, it is called the **Cauchy principal value** of the improper integral.

Problem 22.5: Find the Cauchy principal value of

$$\int_{-4}^5 3/x^3 dx.$$

INTRODUCTION TO CALCULUS

MATH 1A

Unit 23: PDF and CDF

LECTURE

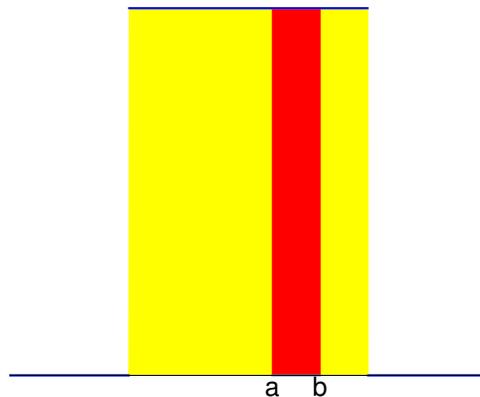
23.1. In **probability theory** one considers functions too:

Definition: A non-negative piece-wise continuous function $f(x)$ which has the property that $\int_{-\infty}^{\infty} f(x) dx = 1$ is called a **probability density function**. For every interval $A = [a, b]$, the number

$$P[A] = \int_a^b f(x) dx$$

is the **probability** of the event.

23.2. An important case is the function $f(x)$ which is 1 on the interval $[0, 1]$ and 0 else. It is the **uniform distribution** on $[0, 1]$. Random number generators in computers first of all generate random numbers with that distribution. In Mathematica, you get such numbers by evaluating **Random[]**. In Python you get it with **import random; random.uniform(0,1)**. The probability $\int_{0.3}^{0.7} f(x) dx$ for example is 0.4. Here is the function $f(x)$:

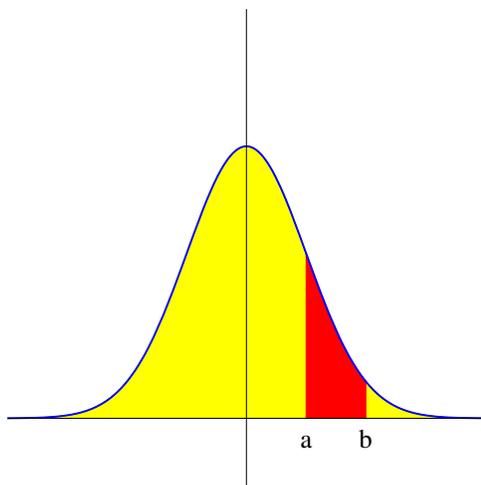


23.3. An other important probability density is the **standard normal distribution**, also called **Gaussian distribution**.

Definition: The **normal distribution** has the density

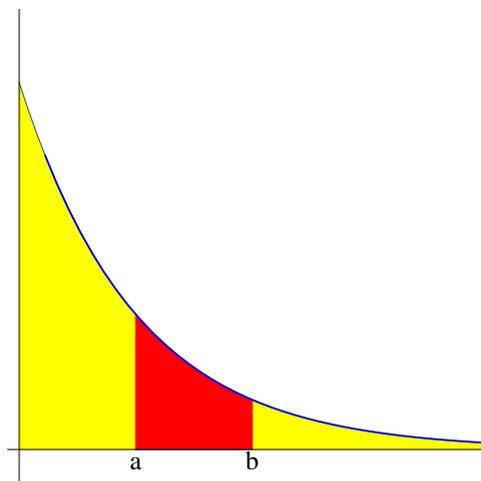
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} .$$

23.4. It is the distribution which appears most often if data can take both positive and negative values. One reason why it appears so often is that if one observes different unrelated quantities then their sum, suitably normalized is close to the normal distribution. **Errors** for example often have normal distribution. Astronomers like Galileo noticed this already in 1630ies. Laplace in 1774 first defined probability distributions and Gauss in 1801 first looked at the normal distribution, also in the context of analyzing astronomical data when searching for the **dwarf planet** Ceres.



Example: The probability density function of the **exponential distribution** is defined as $f(x) = e^{-x}$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$. It is used to measure lengths of arrival times like the time until you get the next email. The density is zero for negative x because there is no way we can travel back in time.

What is the probability that you get an email between times $x = 1$ and times $x = 2$? Answer: it is $\int_1^2 f(x) dx = e^{-1} - e^{-2} = 1/e - 1/e^2$.



Definition: Assume f is a probability density function (PDF). The anti-derivative $F(x) = \int_{-\infty}^x f(t) dt$ is called the **cumulative distribution function** (CDF).

Example: For the exponential function the cumulative distribution function is

$$\int_{-\infty}^x f(x) dx = \int_0^x f(x) dx = -e^{-x} \Big|_0^x = 1 - e^{-x} .$$

Definition: The probability density function $f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$ is called the **Cauchy distribution**.

Example: Find the cumulative distribution function of the Cauchy distribution.

Solution:

$$F(x) = \int_{-\infty}^x f(t) dt = \frac{1}{\pi} \arctan(x) \Big|_{-\infty}^x = \left(\frac{1}{\pi} \arctan(x) + \frac{1}{2} \right) .$$

Definition: The **mean** of a distribution is the number

$$m = \int_{-\infty}^{\infty} x f(x) dx .$$

Example: The mean of the distribution $f(x) = e^{-x}$ on $[0, \infty)$ is

$$\int_0^{\infty} x e^{-x} dx .$$

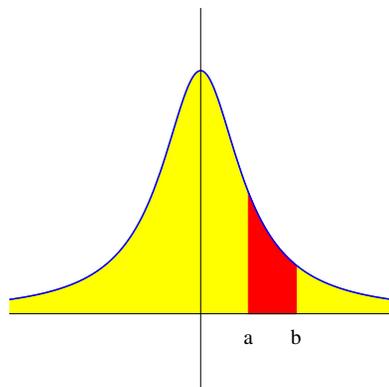
We do not know yet how to compute this but learn a technique later. For now, we have to guess the anti derivative or being told that it is $(-1 - x)e^{-x}$. We can check that the derivative of this function is indeed e^{-x} . So,

$$\int_0^{\infty} x e^{-x} dx = \lim_{t \rightarrow \infty} (-1 - x) e^{-x} \Big|_0^t = \lim_{t \rightarrow \infty} (-1 - t) e^{-t} + 1 = 1 .$$

23.5. The distribution looks similar to the Gaussian distribution, but it has more risk. The **variance** of this distribution

$$\int_{-\infty}^{\infty} x^2 f(x) dx = (1/\pi) \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dx$$

is infinite. The function $\frac{x^2}{1+x^2}$ is asymptotically 1 and has a divergent integral from $-\infty$ to ∞ .



Homework

Problem 23.1: Assume the probability density for the time you have to wait for your next text message you get is $f(x) = 5e^{-5x}$ where x is time in hours.

What is the probability that you get your next text message in the next 4 hours but not before 1 hour?

Problem 23.2: Assume the probability distribution for the waiting time to the next warm day is $f(x) = (1/4)e^{-x/4}$, where x has days as unit.

What is the probability to get a warm day between tomorrow and after tomorrow that is between $x = 1$ and $x = 2$?

Problem 23.3: Verify that the function $f(x)$ which is defined to be zero outside the interval $[-1, 1]$ and given as $\frac{1}{\pi\sqrt{1-x^2}}$ inside the interval $[-1, 1]$ is a probability distribution.

What is the cumulative distribution function?

Problem 23.4: Assume some risky experiment leads to discrepancies (errors) which are distributed according to the Cauchy distribution.

- Find the probability that the error is in absolute value larger than 1.
- Find the probability that the error is smaller than $-\sqrt{3}/2$.

Problem 23.5: If $f(x)$ is a probability distribution, then $\int_{-\infty}^{\infty} xf(x) dx$ is called the **mean** of the distribution.

- Compute the mean for the standard normal distribution.
- Compute the mean for the Cauchy distribution $f(x) = \frac{1}{\pi} \frac{1}{1+x^2}$.
- Compute the mean for the arc-sin distribution $f(x) = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}}$ on $[-1, 1]$.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 24: Substitution

LECTURE

24.1. We know how to integrate functions like e^{6x} or $1/(1+x)$. The technique of **substitution** allows to spot more complicated anti-derivatives. If we differentiate the function $\sin(x^2)$ and use the chain rule, we get $\cos(x^2)2x$. Indeed, by the fundamental theorem of calculus, the anti-derivative of $\cos(x^2)2x$ is

$$\int \cos(x^2)2x \, dx = \sin(x^2) + C .$$

24.2. How can we see the integral without knowing the result already? Here is a very important case:

If $f(x) = g(u(x))u'(x)$, then the anti-derivative of f is $G(u(x)) + C$, where G is the anti-derivative of g .

Example: Find the anti derivative of

$$f(x) = e^{x^4+x^2}(4x^3 + 2x) .$$

Solution: It is $e^{x^4+x^2} + C$.

Example: Find

$$\int \sqrt{x^5 + 1}x^4 \, dx .$$

Solution. Try $(x^5+1)^{3/2}$ and differentiate. This gives $15/2$ of what we have. Therefore $F(x) = (2/15)(x^5 + 1)^{3/2}$.

Example: Find the anti derivative of

$$\frac{\log(x)}{x} .$$

Solution: We spot that $1/x$ is the derivative of $\log(x)$. The anti-derivative is $\log(x)^2/2 + C$.

24.3. Writing down the function and adjusting a constant is the “speedy rule”:

If $\int f(ax + b) dx = F(ax + b)/a$ where F is the anti derivative of f .



Example: $\int \sqrt{x+1} dx$. **Solution:** $(x+1)^{3/2}(2/3) + C$.

Example: $\int \frac{1}{1+(5x+2)^2} dx$. **Solution:** $\arctan(5x+2)(1/5) + C$.

24.4. The method of substitution formalizes this: A) select part of the formula, call it u . B) then write $du = u'dx$. C) replace dx with du/u' . D) If all terms x have disappeared, integrate. E) Back substitute the variable x . If things should not work, go back to A) and try an other u .

$$\int f(u(x)) u'(x) dx = \int g(u) du .$$

24.5. We aim to end up with an integral $\int g(u) du$ which does not involve x anymore. Finally, after integration of this integral, do a **back-substitution**: replace the variable u again with the function $u(x)$.

Example: Find the anti-derivative of $\int \log(x)/x dx$. **Solution:** Pick $u = \log(x)$, $du = (1/x)dx$. Because $dx = xdu$, we get $\int udu = u^2/2 + C$. Back substitute to get $\log^2(x)/2 + C$.

Example: Find the anti-derivative

$$\int \log(\log(x)) \frac{1}{\log(x)x} dx .$$

Solution: Try $u = \log(x)$ and $du = (1/x)dx$, then plug this into the formula. It gives $\int \log(u)/u du$. We have just solved this integral before and got $\log(u)^2/2 + C$.

Example: Solve the integral

$$\int \frac{x}{1+x^4} dx .$$

Solution: Substitute $u = x^2$, $du = 2xdx$ gives $(1/2) \int du/(1+u^2) du = (1/2) \arctan(u) = (1/2) \arctan(x^2) + C$.

Example: What is the anti-derivative of $\sin(\sqrt{x})/\sqrt{x}$?

Solution. Try $u = \sqrt{x}$, $x = u^2$, $dx = 2udu$. The result is $-2 \cos(\sqrt{x}) + C$.

24.6. Here is an example that is more challenging

Example: Solve the integral

$$\int \frac{x^3}{\sqrt{x^2+1}} dx .$$

Solution. Trying $u = \sqrt{x^2+1}$ does not work. Try $u = x^2 + 1$, then $du = 2xdx$ and $dx = du/(2\sqrt{u-1})$. Substitute this in to get

$$\int \frac{\sqrt{u-1}^3}{2\sqrt{u-1}\sqrt{u}} du = \int \frac{(u-1)}{2\sqrt{u}} = \int u^{1/2}/2 - u^{-1/2}/2 du = u^{3/2}/3 - u^{1/2} = \frac{(x^2+1)^{3/2}}{3} - (x^2+1)^{1/2} .$$

24.7. When doing **definite integrals** $\int_a^b f(x) dx$, we could find the anti-derivative as described and then fill in the boundary points. Substituting the boundaries directly accelerates the process since we do not have to substitute back to the original variables:

$$\int_a^b g(u(x))u'(x) dx = \int_{u(a)}^{u(b)} g(u) du .$$

Proof. This identity follows from the fact that the right hand side is $G(u(b)) - G(u(a))$ by the fundamental theorem of calculus. The integrand on the left has the anti derivative $G(u(x))$. Again by the fundamental theorem of calculus the integral leads to $G(u(b)) - G(u(a))$.

Example: Find the anti-derivative of $\int_0^2 \sin(x^3 - 1)x^2 dx$. **Solution:**

$$\int_{x=0}^{x=2} \sin(x^3 + 1)x^2 dx .$$

Solution: Use $u = x^3 + 1$ and get $du = 3x^2dx$. We get

$$\int_{u=1}^{u=9} \sin(u)du/3 = (1/3) \cos(u)|_1^9 = [-\cos(9) + \cos(1)]/3 .$$

Example: $\int_0^1 \frac{1}{5x+1} dx = [\log(u)]/5|_1^6 = \log(6)/5$.

Example: $\int_3^5 \exp(4x - 10) dx = [\exp(10) - \exp(2)]/4$.

24.8. Substituting the bounds can sometimes be a bit tricky. An alternative way is to find first the anti derivative and then plug in the original bounds. Avoiding substituting the bounds actually is often the preferred way.

Example: $\int_3^5 \exp(4x - 10) dx = F(5) - f(3)$, where $F(x) = \exp(4x - 10)/4$.

Homework

Problem 24.1: Find the following anti-derivatives.

- a) $\int 20x \sin(x^2) dx$
- b) $\int e^{x^6+x} (6x^5 + 1) dx$
- c) $\cos(\cos^3(x)) \sin(x) \cos^2(x)$
- d) $e^{\tan(x)} / \cos^2(x)$.

Problem 24.2: Compute the following definite integrals. It is fine to find first the anti-derivative and only in the end place the bounds:

- a) $\int_2^5 \sqrt{x^5 + x} (x^4 + 1/5) dx$
- b) $\int_0^{\sqrt{\pi}} \sin(x^2) x dx$.
- c) $\int_{1/e}^e \frac{\sqrt{\log(x)}}{x} dx$.
- d) $\int_0^1 \frac{5x}{\sqrt{1+x^2}} dx$.

Problem 24.3: Find the definite integral

$$\int_e^{6e} \frac{dx}{\sqrt{\log(x)} x}.$$

Problem 24.4: a) Find the indefinite integral

$$\int \frac{x^5}{\sqrt{x^2 + 1}} dx.$$

b) Find the anti-derivative of

$$f(x) = \frac{1}{x(1 + \log(x)^2)}.$$

Problem 24.5: This is again a more challenging example, similar as we did in class. You already did work on it in 24.4a). Now compute the definite integral

$$\int_0^1 \frac{x^5}{\sqrt{x^2 + 1}} dx$$

INTRODUCTION TO CALCULUS

MATH 1A

Unit 25: Integration by parts

25.1. Integrating the product rule $(uv)' = u'v + uv'$ gives the method **integration by parts**. It complements the method of substitution we have seen last time. As a rule of thumb, always try first to **1) simplify a function and integrate using known functions**, then **2) try substitution** and finally **3) try integration by parts**.

$$\int u(x) v'(x) dx = u(x)v(x) - \int u'(x)v(x) dx.$$

Example: To see how integration by parts work, lets try to find $\int x \sin(x) dx$. First identify what you want to differentiate and call it u , the part to integrate is called v' . Now, write down uv and subtract a new integral which integrates $u'v$:

$$\int x \sin(x) dx = x (-\cos(x)) - \int 1 (-\cos(x)) dx = -x \cos(x) + \sin(x) + C dx .$$

In class, I will will stream line this by just placing an arrow down under the expression you differentiate and an arrow up under the expression you integrate. You remember to first integrate, then subtract the integral of the expression where you both integrate and differentiate. If you like to write down the u, v , do so and remember

$$\int u dv = uv - \int v du .$$

Example: Find $\int x e^x dx$. **Solution.** You want to differentiate x and integrate e^x .

$$\int x \exp(x) dx = x \exp(x) - \int 1 \cdot \exp(x) dx = x \exp(x) - \exp(x) + C dx .$$

Example: Find $\int \log(x) dx$. **Solution.** While there is only one function here, we need two to use the method. Let us look at $\log(x) \cdot 1$:

$$\int \log(x) 1 dx = \log(x)x - \int \frac{1}{x} x dx = x \log(x) - x + C .$$

Example: Find $\int x \log(x) dx$. **Solution.** Since we know from the previous problem how to integrate log we could proceed by taking $x = u$. We can also take $u = \log(x)$ and $dv = x$:

$$\int \log(x) x dx = \log(x) \frac{x^2}{2} - \int \frac{1}{x} \frac{x^2}{2} dx$$

which is $\log(x)x^2/2 - x^2/4$.

25.2. We see that it is often better to differentiate log first. The word LIATE explained below tells which functions we want to call u and differentiate.

Example: Marry go round: Find $I = \int \sin(x) \exp(x) dx$. **Solution.** Lets integrate $\exp(x)$ and differentiate $\sin(x)$.

$$= \sin(x) \exp(x) - \int \cos(x) \exp(x) dx .$$

Lets do it again:

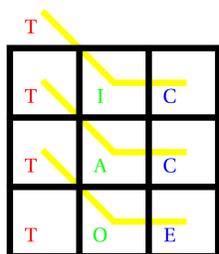
$$= \sin(x) \exp(x) - \cos(x) \exp(x) - \int \sin(x) \exp(x) dx .$$

We moved in circles and are stuck! But wait. Are we really? We have now derived an identity

$$I = \sin(x) \exp(x) - \cos(x) \exp(x) - I$$

which we can solve for I and get $I = [\sin(x) \exp(x) - \cos(x) \exp(x)]/2$.

Tic-Tac-Toe



Integration by parts can become complicated if it has to be done several times. Keeping the order of the signs can be especially daunting. Fortunately, there is a powerful **tabular integration by parts method**. It has been called “**Tic-Tac-Toe**” in the movie Stand and deliver. Lets call it **Tic-Tac-Toe** therefore.

Example: Find the anti-derivative of $(x - 1)^3 e^{2x}$. **Solution:**

$(x - 1)^3$	$\exp(2x)$	
$3(x - 1)^2$	$\exp(2x)/2$	\oplus
$6(x - 1)$	$\exp(2x)/4$	\ominus
6	$\exp(2x)/8$	\oplus
0	$\exp(2x)/16$	\ominus

The anti-derivative is

$$(x - 1)^3 e^{2x} / 2 - 3(x - 1)^2 e^{2x} / 4 + 6(x - 1) e^{2x} / 8 - 6e^{2x} / 16 + C .$$

Example: Find the anti-derivative of $x^2 \cos(x)$. **Solution:**

x^2	$\cos(x)$	
$2x$	$\sin(x)$	\oplus
2	$-\cos(x)$	\ominus
0	$-\sin(x)$	\oplus

The anti-derivative is $x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + C$.

Example: More extreme Find the anti-derivative of $x^7 \cos(x)$. **Solution:**

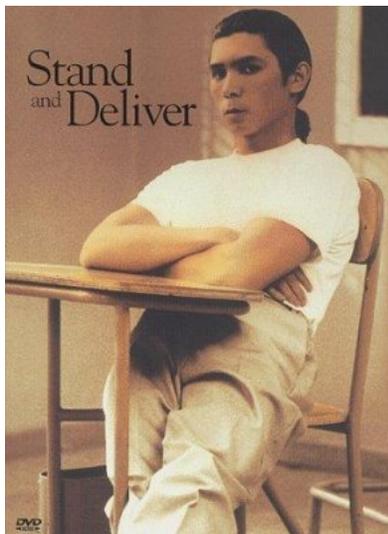
x^7	$\cos(x)$	
$7x^6$	$\sin(x)$	\oplus
$42x^5$	$-\cos(x)$	\ominus
$120x^4$	$-\sin(x)$	\oplus
$840x^3$	$\cos(x)$	\ominus
$2520x^2$	$\sin(x)$	\oplus
$5040x$	$-\cos(x)$	\ominus
5040	$-\sin(x)$	\oplus
0	$\cos(x)$	\ominus

The anti-derivative is

$$\begin{aligned}
 F(x) &= x^7 \sin(x) + 7x^6 \cos(x) - 42x^5 \sin(x) \\
 &- 210x^4 \cos(x) + 840x^3 \sin(x) + 2520x^2 \cos(x) \\
 &- 5040x \sin(x) - 5040 \cos(x) + C .
 \end{aligned}$$

25.3. Do this without this method and you see the value of the method.

¹ ² ³.



I myself learned the method from the movie “Stand and Deliver”, where **Jaime Escalante** of the Garfield High School in LA uses the method. It can be traced down to an article of V.N. Murty. The method realizes in a clever way an iterated integration by parts method:

$$\begin{aligned}
 \int f g dx &= f g^{(-1)} - f^{(1)} g^{-2} + f^{(2)} g^{(-3)} - \dots \\
 &- (-1)^n \int f^{(n+1)} g^{(-n-1)} dx
 \end{aligned}$$

The method can be verified by induction because the f function is differentiated again and again and the g function is integrated again and again. The alternating minus-plus-signs come from the fact that we subtract again an integral. We always pair a k 'th derivative with a $k + 1$ 'th integral and take the sign $(-1)^k$.

Coffee or Tea?

¹V.N. Murty, Integration by parts, Two-Year College Mathematics Journal 11, 1980, p. 90-94.

²D. Horowitz, Tabular Integration by Parts, College Mathematics Journal, 21, 1990, p. 307-311.

³K.W. Folley, integration by parts, American Mathematical Monthly 54, 1947, p. 542-543

25.4. When doing integration by parts, We want to try first to differentiate **L**ogs, **I**nverse trig functions, **P**owers, **T**rig functions and **E**xponentials. This can be remembered as **LIPTE** which is close to "lipton" (the tea).

For coffee lovers, there is an equivalent one: **L**ogs, **I**nverse trig functions, **A**lgebraic functions, **T**rig functions and **E**xponentials which can be remembered as **LIATE** which is close to "latte" (the coffee).

Whether you prefer to remember it as a "coffee latte" or a "lipton tea" is up to you.

There is even a better method, the "method of the opportunist":

Just integrate what you can integrate and differentiate the rest.

And don't forget to consider integrating 1, if nothing else works.



LIATE



LIPTE



Homework

Problem 25.1: Integrate $\int x^3 \log(x) dx$.

Problem 25.2: Integrate $\int x^5 \sin(x) dx$

Problem 25.3: Find the anti derivative of $\int 2x^6 \exp(x) dx$. (*)

Problem 25.4: Find the anti derivative of $\int \sqrt{x} \log(x) dx$.

Problem 25.5: Find the anti derivative of $\int \sin(x) \exp(-x) dx$.

(*) If you want to get into Guinness see, whether you can integrate $x^{1000} \exp(x)$. Unfortunately, such an entry does not exist as you can see on the website:
<https://guinnessworldrecords.com/search?term=mathematics>.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 26: Review

SCOPE

The second midterm covers units 16-25: catastrophes, integrals, fundamental theorem of calculus and anti-derivatives, areas, volumes, improper integrals, probability distributions, substitution and integration by parts.

MAIN POINTS

Important integration techniques: **use substitution**, (write down u, du, dx) **use integration by parts** ($\int u dv = uv - \int v du$).

Catastrophes are parameter values where the number of minima changes. To find the parameter, look where $f''(x_c)$ becomes zero at the critical point x_c .

Definite integrals $F(x) = \int_0^x f(t) dt$ were defined as a limit of Riemann sums.

A function $F(x)$ satisfying $F' = f$ is called the anti-derivative of f . The general anti-derivative is $F + c$ where c is a constant.

The **fundamental theorem of calculus** tells $d/dx \int_0^x f(x) dx = f(x)$ and $\int_0^x f'(x) dx = f(x) - f(0)$.

The integral $\int_a^b g(x) - f(x) dx$ is the **signed area between the graphs** of f and g . Places, where $f < g$ are counted negative. When area is asked, split things up.

The integral $\int_a^b A(x) dx$ is a **volume** if $A(x)$ is the area of a slice of the solid perpendicular to a point x on an axes.

Write **improper integrals** as limits of definite integrals $\int_1^\infty f(x) dx = \lim_{R \rightarrow \infty} \int_1^R f(x) dx$. We similarly treat points, where f is discontinuous.

One can use CDF's F to compute probabilities: $F(b) - F(a) = \int_a^b f(x) dx$.

One can compute **area, volume, cumulative distribution functions** using integrals.

To determine the **catastrophes** for a family $f_c(x)$ of functions, determine the critical points in dependence of c and find values c , where a critical point changes from a local minimum to a local maximum.

Important integrals

Which one is the derivative which the integral?

$\sin(x)$	$-\cos(x)$.	$\log(x)$	$x \log(x) - x$
$\tan(x)$	$1/\cos^2(x)$.	$1/x$	$\log(x)$
$\arctan(x)$	$1/(1+x^2)$.	$-1/(1+x^2)$	$\operatorname{arccot}(x)$
$1/\sqrt{1-x^2}$	$\arcsin(x)$	$-1/\sqrt{1-x^2}$	$\arccos(x)$

Improper integrals

$\int_1^\infty 1/x^2 dx$ Prototype of improper integral which exists.

$\int_1^\infty 1/x dx$ Prototype of improper integral which does not exist.

$\int_0^1 1/x dx$ Prototype of improper integral which does not exist.

$\int_0^1 1/\sqrt{x} dx$ Prototype of improper integral which does exist.

The fundamental theorem

$$\frac{d}{dx} \int_0^x f(t) dt = f(x)$$

$$\int_0^x f'(t) dt = f(x) - f(0).$$

This implies

$$\int_a^b f'(x) dx = f(b) - f(a)$$

Without limits of integration, we call $\int f(x) dx$ the **anti derivative**. It is defined up to a constant. For example $\int \sin(x) dx = -\cos(x) + C$.

PDF and CDF

Calculus applies directly if there are situations where one quantity is the derivative of the other.

function	anti derivative
probability density function	cumulative distribution function

Most important integrals

The most important integral is the integral

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

holds for all n different from 1.

$$\int \frac{1}{x} dx = \log(x)$$

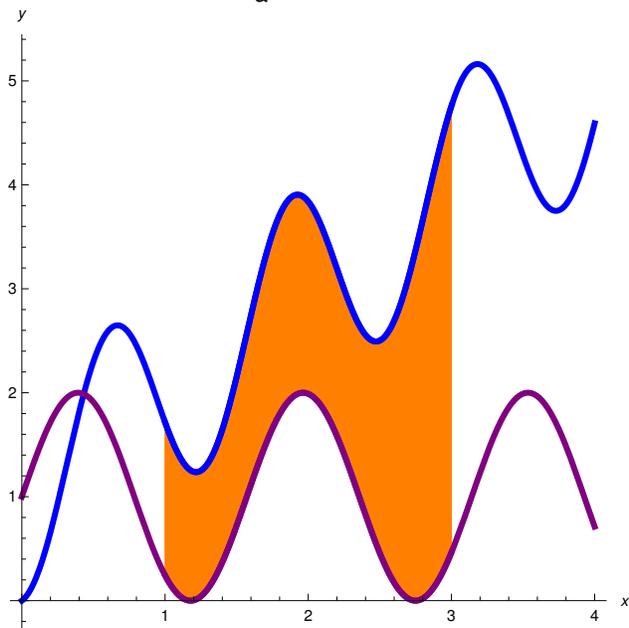
Example: $\int \sqrt{x+7} dx = \frac{2}{3}(x+7)^{3/2}$.

Example: $\int \frac{1}{x+5} dx = \log(x+5)$

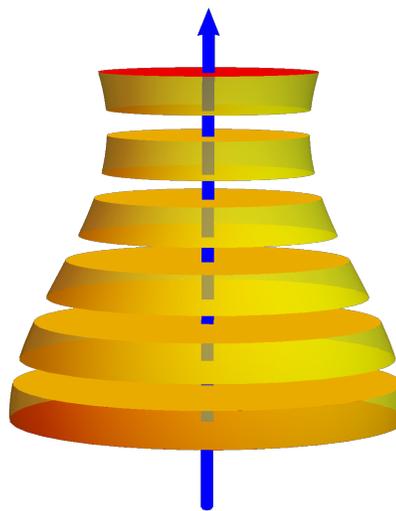
Example: $\int \frac{1}{4x+3} dx = \log(4x+3)/4$

Key pictures

$$\boxed{\text{Area}} = \int_a^b (g(x) - f(x)) \, dx$$

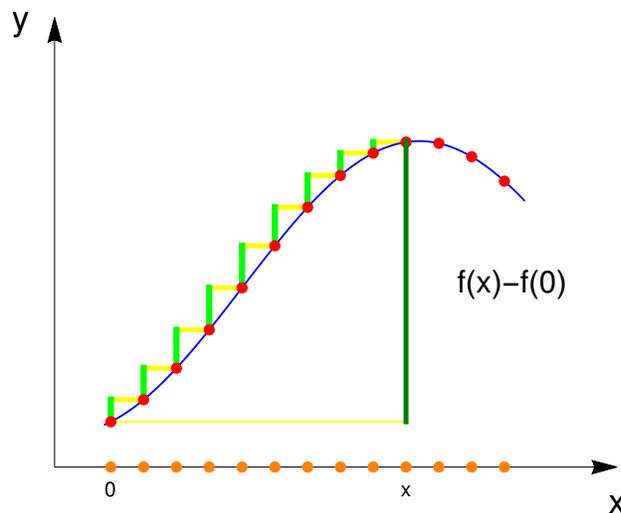
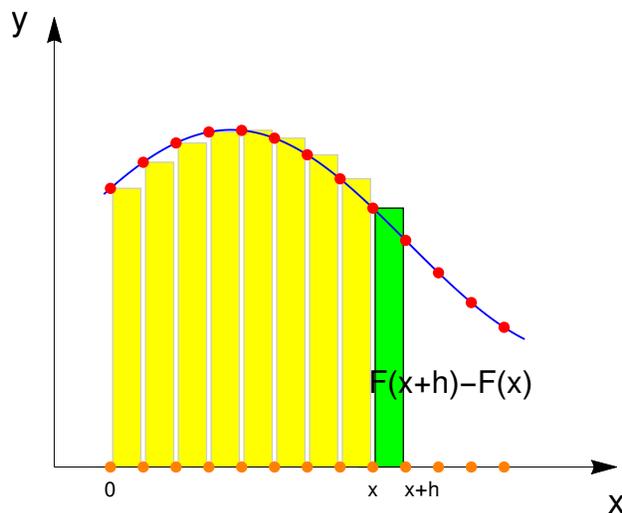


$$\boxed{\text{Volume}} = \int_a^b A(z) \, dz$$



$$\boxed{f(x)} = \frac{d}{dx} \int_0^x f(t) \, dt$$

$$\boxed{f(x) - f(0)} = \int_0^x f'(t) \, dt$$



Make a picture, whenever we deal with an area or volume computation!
 For volume computations, just integrate area of the cross section $A(x)$.
 For area computations integrate $g(x) - f(x)$, where $g(x)$ is above $f(x)$.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 27: Numerical integration

LECTURE

27.1. We look here at some numerical techniques for computing integrals. There are variations of basic Riemann sums but allow speed up or adjust the computation to more complex situations.

Riemann sum with nonuniform spacing

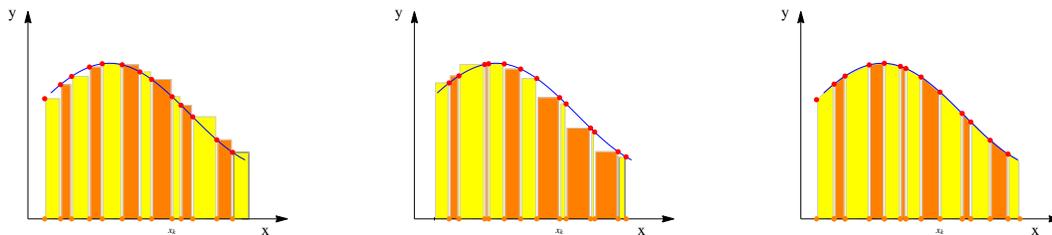
Definition: A more general Riemann sum is obtained by choosing n points $\{x_j\}$ in $[a, b]$ and then to look at

$$S_n = \sum_{y_j} f(x_j)(x_{j+1} - x_j) = \sum_{y_j} f(x_j)\Delta x_j,$$

where $\Delta x_j = x_{j+1} - x_j$.

27.2. This generalization of Riemann and allows to use a small mesh size where the function fluctuates a lot. The function $f(x) = \sin(1/(x^2 + 0.1))$ for example fluctuates near the origin more. We need more division points there.

Definition: The sum $L = \sum f(x_j)\Delta x_j$ is called the **left Riemann sum**, the sum $R = \sum f(x_{j+1})\Delta x_j$ the **right Riemann sum**.



If $x_0 = a, x_n = b$ and $\max_j \Delta x_j \rightarrow 0$ for $n \rightarrow \infty$ then S_n converges to $\int_a^b f(x) dx$.

Example: If $x_j - x_k = 1/n$ and $z_j = x_j$, then we have the Archimedes sum defined initially.

Example: You numerically integrate $\sin(x)$ on $[0, \pi/2]$ with a Riemann sum. What is better, the left Riemann sum or the right Riemann sum? Look also at the interval $[\pi/2, \pi]$? **Solution:** you see that in the first case, the left Riemann sum is smaller than the actual integral. In the second case, the left Riemann sum is larger than the actual integral.

Trapezoid rule

Definition: The average $(L + R)/2$ between the left and right hand Riemann sum is called the **Trapezoid rule**. Geometrically, it sums up areas of trapezoids instead of rectangles.

27.3. The trapezoid rule does not change things in the Archimedes case. For the interval $[0, 1]$ for example, with $x_k = k/n$ we have

$$S_n = \frac{1}{2n}[f(0) + f(1)] + \frac{1}{n} \sum_{k=1}^{n-1} f(x_k).$$

Simpson rule

Definition: The **Simpson rule** computes the sum

$$S_n = \frac{1}{6n} \sum_{k=1}^n [f(x_k) + 4f(y_k) + f(x_{k+1})],$$

where $y_k = (x_k + x_{k+1})/2$ is the midpoint between x_k and x_{k+1} .

27.4. The Simpson rule is exact for quadratic functions: for $f(x) = ax^2 + bx + c$, the formula

$$\frac{1}{v-u} \int_u^v f(x) dx = [f(u) + 4f((u+v)/2) + f(v)]/6$$

holds exactly. To prove it just run the following two lines in Mathematica: (== means "is equal")

```
f[x_] := a x^2 + b x + c;
(f[u]+f[v]+4 f[(u+v)/2])/6 == Integrate[f[x], {x, u, v}]/(v-u)
Simplify[%]
```

27.5. With a bit more calculus one can show that if f is 4 times differentiable then the Simpson rule is n^{-4} close to the actual integral. For 100 division points, this can give accuracy to 10^{-8} already.

There are other variants which are a bit better but need more function values. If x_k, y_k, z_k, x_{k+1} are equally spaced, then

Definition: The **Simpson 3/8 rule** computes

$$\frac{1}{8n} \sum_{k=1}^n [f(x_k) + 3f(y_k) + 3f(z_k) + f(x_{k+1})].$$

This formula is again exact for quadratic functions: for $f(x) = ax^2 + bx + c$, the formula

$$\frac{1}{v-u} \int_u^v f(x) dx = [f(u) + 3f((2u+v)/3) + 3f((u+2v)/3) + f(v)]/6$$

holds. Just run the two Mathematica lines to check this:

```
f[x_] := a x^2 + b x + c; L=Integrate[f[x],{x,u,v}]/(v-u);
Simplify[(f[u]+f[v]+3f[(2u+v)/3]+3f[(u+2v)/3])/8==L]
```

This **Simpson 3/8 method** can be slightly better than the first Simpson rule.

Monte Carlo Method

27.6. A powerful integration method is to choose n random points x_k in $[a, b]$ and look at the sum divided by n . Because it uses randomness, it is called **Monte Carlo method**.

Definition: The **Monte Carlo** integral is the limit S_n to infinity

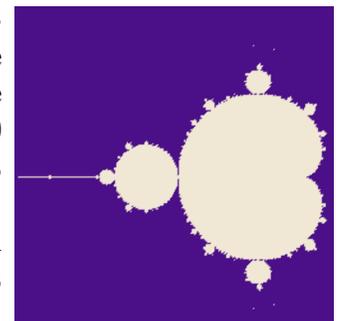
$$S_n = \frac{(b-a)}{n} \sum_{k=1}^n f(x_k),$$

where x_k are n random values in $[a, b]$.

27.7. The law of large numbers in probability shows that the **Monte Carlo integral** is equivalent to the **Lebesgue integral** which is more powerful than the Riemann integral. Monte Carlo integration is interesting especially if the function is complicated.

The following two lines evaluate the **area of the Mandelbrot fractal** using Monte Carlo integration. The function F is equal to 1, if the parameter value c of the quadratic map $z \rightarrow z^2 + c$ is in the Mandelbrot set and 0 else. It shoots 100'000 random points and counts what fraction of the square of area 9 is covered by the set. Numerical experiments give values close to the actual value around 1.51.... One could use more points to get more accurate estimates.

Example:



```
F[c_] := Block[{z=c, u=1}, Do[z=N[z^2+c]; If[Abs[z]>3, u=0; z=3], {99}]; u];
M=10^5; Sum[F[-2.5+3 Random[]+I(-1.5+3 Random[])], {M}]* (9.0/M)
```

Homework

Problem 27.1: Use the generalized left Riemann sum with $x_0 = 0$, $x_1 = \pi/6$, $x_2 = \pi/2$, $x_3 = 2\pi/3$ and $x_4 = \pi$ to compute the integral

$$\int_0^{\pi} \sin(x) dx$$

without a computer.

Problem 27.2: Use a computer to generate 10 random numbers x_k in $[0, 1]$. If you do not have a computer to do that for you, make up some random numbers on your own. Try to be as random as possible. Sum up the cubes x_k^3 of these numbers and divide by 10. Compare your result with $\int_0^1 x^3 dx$.

Remark. If using a program, increase the value of n as large as you can. Here is a Mathematica code:

```
n=10; Sum[ Random[]^3 , { n } ]/n
```

Problem 27.3: Use the Simpson rule to compute $\int_0^{\pi} \sin(x) dx$ using $n = 2$ intervals $[a, b] = [0, \pi/2]$ or $[a, b] = [\pi/2, \pi]$. On each of these two intervals $[a, b]$, compute the Simpson value

$$\frac{[f(a) + 4f((a+b)/2) + f(b)]}{6}(b-a)$$

with $f(x) = \sin(x)$ then add up. Compare with the actual integral.

Problem 27.4: Now use the 3/8 Simpson rule to estimate $\int_0^{\pi} \sin(x) dx$ using $n = 1$ intervals $[0, \pi]$. Again compare with the actual integral.

Problem 27.5: a) Use a computer to numerically integrate

$$\int_0^1 \sin\left(\frac{1}{x^2}\right) \frac{1}{x^2} dx .$$

b) Do the same with

$$\int_0^1 \sin\left(\frac{1}{x^2}\right) \frac{1}{x^4} dx .$$

INTRODUCTION TO CALCULUS

MATH 1A

Unit 28: Partial fractions

LECTURE

The method of partial fractions is not really about integration. It is about algebra. We have learned how to integrate polynomials like $x^4 + 5x + 3$. What about rational functions? We will see here that they are a piece of cake if you know a bit about algebra.



28.1. Lets see what we know already:

- We also know that integrating $1/x$ gives $\log(x)$. We can for example integrate

$$\int \frac{1}{x-6} dx = \log(x-6) + C .$$

- We also have learned how to integrate $1/(1+x^2)$. It was an important integral:

$$\int \frac{1}{1+x^2} dx = \arctan(x) + C .$$

Using substitution, we can do more like

$$\int \frac{dx}{1+4x^2} = \int \frac{du/2}{1+u^2} = \arctan(u)/2 = \arctan(2x)/2 .$$

- We also know how to integrate functions of the type $x/(x^2+c)$ using substitution. We can write $u = x^2 + c$ and get $du = 2xdx$ so that

$$\int \frac{x}{x^2+c} dx = \int \frac{1}{2u} du = \frac{\log(x^2+c)}{2} .$$

- Also functions $1/(x+c)^2$ can be integrated using substitution. With $x+c = u$ we get $du = dx$ and

$$\int \frac{1}{(x+c)^2} dx = \int \frac{1}{u^2} du = -\frac{1}{u} + C = -\frac{1}{x+c} + C .$$

28.2. We would love to be able to integrate any rational function

$$f(x) = \frac{p(x)}{q(x)},$$

where p, q are polynomials. This is where **partial fractions come in**. The idea is to write a rational function as a sum of fractions we know how to integrate. The above examples have shown that we can integrate $a/(x+c)$, $(ax+b)/(x^2+c)$, $a/(x+c)^2$ and cases, which after substitution are of this type.

Definition: The **partial fraction method** writes $p(x)/q(x)$ as a sum of functions of the above type which we can integrate.

28.3. This is an algebra problem. Here is an important special case:

In order to integrate $\int \frac{1}{(x-a)(x-b)} dx$, write

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}.$$

and solve for A, B .

28.4. In order to solve for A, B , write the right hand side as one fraction again

$$\frac{1}{(x-a)(x-b)} = \frac{A(x-b) + B(x-a)}{(x-a)(x-b)}.$$

We only need to look at the nominator:

$$1 = Ax - Ab + Bx - Ba.$$

In order that this is true we must have $A + B = 0$, $Ab - Ba = 1$. This allows us to solve for A, B .

Examples

Example: To integrate $\int \frac{2}{1-x^2} dx$ we can write

$$\frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$$

and integrate each term

$$\int \frac{2}{1-x^2} = \log(1+x) - \log(1-x).$$

Example: Integrate $\frac{5-2x}{x^2-5x+6}$. **Solution.** The denominator is factored as $(x-2)(x-3)$. Write

$$\frac{5-2x}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2}.$$

Now multiply out and solve for A, B :

$$A(x-2) + B(x-3) = 5-2x.$$

This gives the equations $A + B = -2$, $-2A - 3B = 5$. From the first equation we get $A = -B - 2$ and from the second equation we get $2B + 4 - 3B = 5$ so that $B = -1$ and so $A = -1$. We have not obtained

$$\frac{5 - 2x}{x^2 - 5x + 6} = -\frac{1}{x - 3} - \frac{1}{x - 2}$$

and can integrate:

$$\int \frac{5 - 2x}{x^2 - 5x + 6} dx = -\log(x - 3) - \log(x - 2).$$

Actually, we could have got this one also with substitution. How?

Example: Integrate $f(x) = \int \frac{1}{1-4x^2} dx$. **Solution.** The denominator is factored as $(1 - 2x)(1 + 2x)$. Write

$$\frac{A}{1 - 2x} + \frac{B}{1 + 2x} = \frac{1}{1 - 4x^2}.$$

We get $A = 1/4$ and $B = -1/4$ and get the integral

$$\int f(x) dx = \frac{1}{4} \log(1 - 2x) - \frac{1}{4} \log(1 + 2x) + C.$$

28.5. There is a fast method to get the coefficients:

If a is different from b , then the coefficients A, B in

$$\frac{p(x)}{(x - a)(x - b)} = \frac{A}{x - a} + \frac{B}{x - b},$$

are

$$A = \lim_{x \rightarrow a} (x - a)f(x) = p(a)/(a - b), \quad B = \lim_{x \rightarrow b} (x - b)f(x) = p(b)/(b - a).$$

Proof. If we multiply the identity with $x - a$ we get

$$\frac{p(x)}{(x - b)} = A + \frac{B(x - a)}{x - b}.$$

Now we can take the limit $x \rightarrow a$ without peril and end up with $A = p(a)/(a - b)$.

28.6. Cool, isn't it? This **Hospital method** or **residue problem** saves time especially with many functions where we would a complicated system of linear equations would have to be solved.

Math is all about elegance. Avoid complicated methods if simple ones are available.

28.7. Here are examples:

Example: Find the anti-derivative of $f(x) = \frac{2x+3}{(x-4)(x+8)}$. **Solution.** We write

$$\frac{2x+3}{(x-4)(x+8)} = \frac{A}{x-4} + \frac{B}{x+8}$$

Now $A = \frac{2 \cdot 4 + 3}{4 + 8} = 11/12$, and $B = \frac{2 \cdot (-8) + 3}{(-8 - 4)} = 13/12$. We have

$$\frac{2x+3}{(x-4)(x+8)} = \frac{(11/12)}{x-4} + \frac{(13/12)}{x+8}.$$

The integral is

$$\frac{11}{12} \log(x-4) + \frac{13}{12} \log(x+8).$$

Example: Find the anti-derivative of $f(x) = \frac{x^2+x+1}{(x-1)(x-2)(x-3)}$. **Solution.** We write

$$\frac{x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Now $A = \frac{1^2+1+1}{(1-2)(1-3)} = 3/2$ and $B = \frac{2^2+2+1}{(2-1)(2-3)} = -7$ and $C = \frac{3^2+3+1}{(3-1)(3-2)} = 13/2$. The integral is

$$\frac{3}{2} \log(x-1) - 7 \log(x-2) + \frac{13}{2} \log(x-3).$$

HOMEWORK

Problem 28.1: a) $\int \frac{1}{x^2-14x+45} dx$
b) $\int \frac{2}{x^2-9} dx$

Problem 28.2: $\int \frac{5dx}{4x^2+1}$. (Does not need partial fractions!)

Problem 28.3: $\int \frac{x^3-x+1}{x^2-1} dx$.

Problem 28.4: $\int \frac{23}{(x+2)(x-3)(x-2)(x+3)} dx$. Use Hospital.

Problem 28.5: $\int \frac{1}{(x+1)(x-1)(x+7)(x-3)} dx$. Use Hospital.

Hint for 3). Subtract first a polynomial.

INTRODUCTION TO CALCULUS

MATH 1A

Unit 29: Trig Substitution

LECTURE

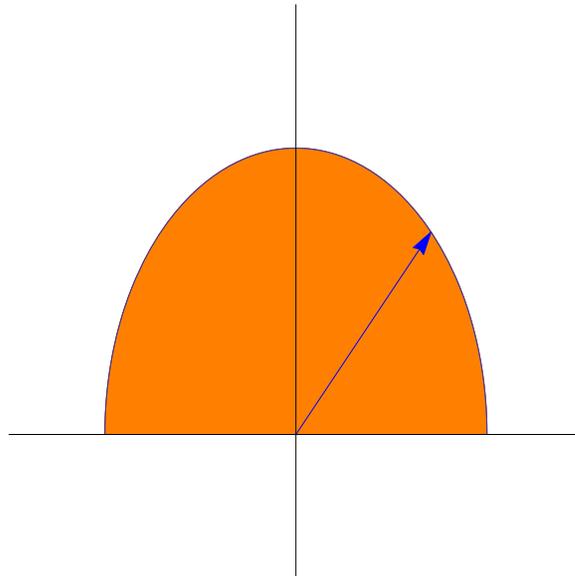
29.1. A **trig substitution** is a special substitution, where x is a trigonometric function of u or u is a trigonometric function of x . Here is an important example:

Example: The area of a half circle of radius 1 is given by the integral

$$\int_{-1}^1 \sqrt{1-x^2} dx .$$

Solution. Write $x = \sin(u)$ so that $\cos(u) = \sqrt{1-x^2}$. $dx = \cos(u)du$. We have $\sin(-\pi/2) = -1$ and $\sin(\pi/2) = 1$ the answer is

$$\int_{-\pi/2}^{\pi/2} \cos(u) \cos(u) du = \int_{-\pi/2}^{\pi/2} (1 + \cos(2u))/2 = \frac{\pi}{2} .$$



29.2. Let us do the same computation for a general radius r :

Example: Compute the area of a half disc of radius r which is given by the integral

$$\int_{-r}^r \sqrt{r^2-x^2} dx .$$

Solution. Write $x = r \sin(u)$ so that $r \cos(u) = \sqrt{r^2 - x^2}$ and $dx = r \cos(u) du$ and $r \sin(-\pi/2) = -r$ and $r \sin(\pi/2) = r$. The answer is

$$\int_{-\pi/2}^{\pi/2} r^2 \cos^2(u) du = r^2 \pi/2 .$$

29.3. Here is an example, we know already how to integrate. But now we derive it from scratch:

Example: Find the integral

$$\int \frac{dx}{\sqrt{1-x^2}} .$$

We know the answer is $\arcsin(x)$. How can we do that without knowing? **Solution.** We can do it also with a trig substitution. Try $x = \sin(u)$ to get $dx = \cos(u) du$ and so

$$\int \frac{\cos(u) du}{\cos(u)} = u = \arcsin(x) + C .$$

29.4. In the next example, $x = \tan(u)$ works. You have to be told that first as it is hard to come up with the idea:

Example: Find the following integral:

$$\int \frac{dx}{x^2 \sqrt{1+x^2}}$$

by using the substitution $x = \tan(u)$. **Solution.** Then $1 + x^2 = 1/\cos^2(u)$ and $dx = du/\cos^2(u)$. We get

$$\int \frac{du}{\cos^2(u) \tan^2(u) (1/\cos(u))} = \int \frac{\cos(u)}{\sin^2(u)} du = -1/\sin(u) = -1/\sin(\arctan(x)) .$$

29.5. For trig substitution, the following basic trig identity is important:

$$\cos^2(u) + \sin^2(u) = 1$$

Depending on whether dividing by $\sin^2(u)$ or $\cos^2(u)$, we get

$$1 + \tan^2(u) = 1/\cos^2(u), \quad 1 + \cot^2(u) = 1/\sin^2(u)$$

These identities come handy: lets look at more examples:

Example: Evaluate the following integral

$$\int x^2/\sqrt{1-x^2} dx .$$

Solution: Substitute $x = \cos(u)$, $dx = -\sin(u) du$ and get

$$\int -\frac{\cos^2(u)}{\sin(u)} \sin(u) du = -\int \cos^2(u) du = -\frac{u}{2} - \frac{\sin(2u)}{4} + C = -\frac{\arcsin(x)}{2} + \frac{\sin(2 \arcsin(x))}{4} + C .$$

Example: Evaluate the integral

$$\int \frac{dx}{(1+x^2)^2}.$$

Solution: we make the substitution $x = \tan(u)$, $dx = du/(\cos^2(u))$. Since $1+x^2 = \cos^{-2}(u)$ we have

$$\int \frac{dx}{(1+x^2)^2} = \int \cos^2(u) du = (u/2) + \frac{\sin(2u)}{4} + C = \frac{\arctan(u)}{2} + \frac{\sin(2 \arctan(u))}{4} + C.$$

29.6. Here is an other prototype problem:

Example: Find the anti derivative of $1/\sin(x)$. **Solution:** We use the substitution $u = \tan(x/2)$ which gives $x = 2 \arctan(u)$, $dx = 2du/(1+u^2)$. Because $1+u^2 = 1/\cos^2(x/2)$ we have

$$\frac{2u}{1+u^2} = 2 \tan(x/2) \cos^2(x/2) = 2 \sin(x/2) \cos(x/2) = \sin(x).$$

Plug this into the integral

$$\int \frac{1}{\sin(x)} dx = \int \frac{1+u^2}{2u} \frac{2du}{1+u^2} = \int \frac{1}{u} du = \log(u) + C = \log(\tan(\frac{x}{2})) + C.$$

Unlike before, where x is a trig function of u , now u is a trig function of x . This example shows that the substitution $u = \tan(x/2)$ is magic. It leads to the following formulas:

$$\begin{aligned} 0. & u = \tan(x/2) \\ \boxed{1} & dx = \frac{2du}{(1+u^2)} \\ \boxed{2} & \sin(x) = \frac{2u}{1+u^2} \\ \boxed{3} & \cos(x) = \frac{1-u^2}{1+u^2} \end{aligned}$$

29.7. It allows us to reduce any rational function involving trig functions to rational functions.

1

Any function $p(x)/q(x)$ where p, q are trigonometric polynomials can now be integrated using elementary functions.

¹Proofs: $\boxed{1}$ differentiate to get $du = dx/(2 \cos^2(x/2)) = dx(1+u^2)/2$. $\boxed{2}$ use double angle $\sin(x) = 2 \tan(x/2) \cos^2(x/2)$ and then $1/\cos^2(x/2) = 1 + \tan^2(x/2)$. $\boxed{3}$ use double angle $\cos(x) = \cos^2(x/2) - \sin^2(x/2) = (1 - \sin^2(x/2)/\cos^2(x/2)) \cos^2(x/2)$ and again $1/\cos^2(x/2) = 1 + \tan^2(x/2)$.

29.8. It is usually a lot of work, but here is an example:

Example: To find the integral

$$\int \frac{\cos(x) + \tan(x)}{\sin(x) + \cot(x)} dx$$

for example, we replace dx , $\sin(x)$, $\cos(x)$, $\tan(x) = \sin(x)/\cos(x)$, $\cot(x) = \cos(x)/\sin(x)$ with the above formulas we get a rational expression which involves u only. This gives us an integral $\int p(u)/q(u) du$ with polynomials p, q . In our case, this would simplify to

$$\int \frac{2u(u^4 + 2u^3 - 2u^2 + 2u + 1)}{(u-1)(u+1)(u^2+1)(u^4 - 4u^2 - 1)} du$$

The method of partial fractions provides us then with the solution.

Homework

Problem 28.1: Find the anti-derivative:

$$\int \sqrt{1-9x^2} dx .$$

Problem 28.2: Find the anti-derivative:

$$\int (1-x^2)^{3/2} dx .$$

Problem 28.3: Find the anti-derivative:

$$\int \frac{\sqrt{1-x^2}}{x^2} dx .$$

Problem 28.4: Integrate

$$\int \frac{dx}{1 + \sin(x)} .$$

Use the substitution $u = \tan(x/2)$ and use the magic box.

Problem 28.5: Compute

$$\int_0^{\pi/3} \frac{dx}{\cos(x)}$$

again using the substitution $u = \tan(x/2)$. Instead of back-substitution, you can also substitute the bounds.

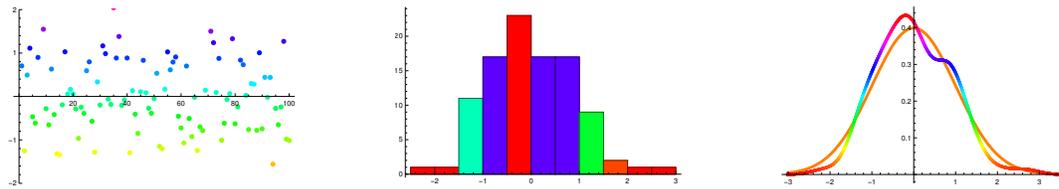
INTRODUCTION TO CALCULUS

MATH 1A

Unit 30: Statistics

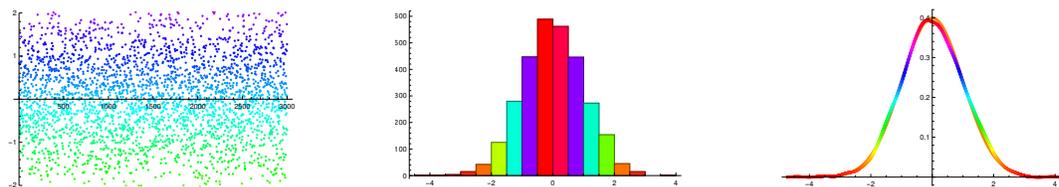
Functions

30.1. Statistics describes the distribution of data using functions. Lets look at the following 100 data points. If we count, how many data values fall into a specific interval, we get a **histogram**. Smoothing this histogram and scaling so that the total integral is 1 produces a **probability distribution function**. This allows us to describe data, discrete sets of points with functions.



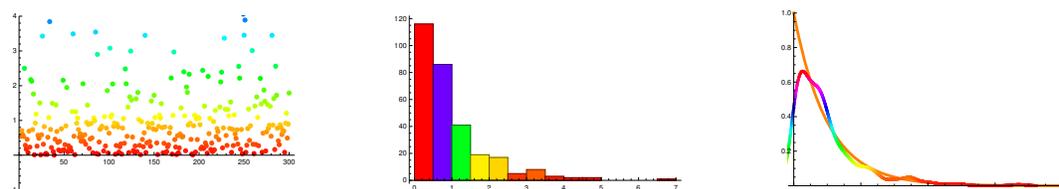
100 data points, the histogram and a smooth interpolation PDF.

30.2. Let us now look at 3000 points. The data distribution has a bell curve shape. In the last picture, we also included the graph of $f(x) = e^{-x^2/2}/\sqrt{2\pi}$.

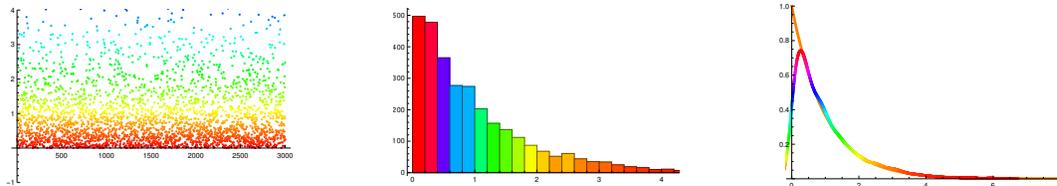


3000 data points, the histogram and a smooth interpolation PDF.

30.3. There are some data of “waiting times. These data are positive. We then draw the histogram and a smooth interpolation function. Lets do it again first for 300 data points and then for 3000 data points.



300 data points, the histogram and a smooth interpolation PDF



3000 other data points, the histogram and a smooth interpolation PDF

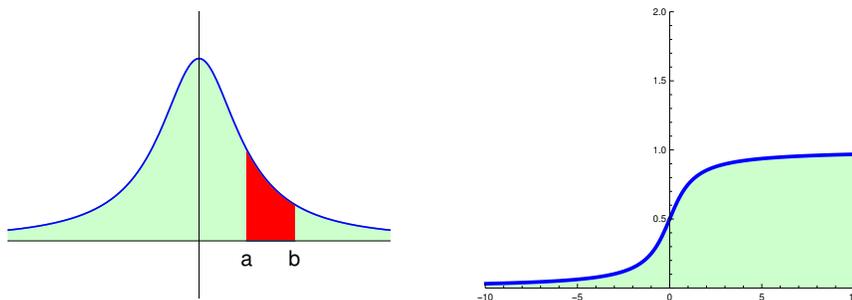
Integration

30.4. We have already defined the probability density function f called PDF and its anti-derivative $F(x)$, the cumulative distribution function CDF.

Definition: Recall that a probability density function is a piecewise continuous function f satisfying $\int_{-\infty}^{\infty} f(x) dx = 1$ and which is ≥ 0 everywhere.

Definition: Of great interest are **moments** of the PDF. These are integrals of the form

$$M_n = \int_{-\infty}^{\infty} x^n f(x) dx$$



PDF and CDF

30.5. For $n = 0$, we know the answer is always $M_0 = 1$. The first moment M_1 is the expectation or average:

Definition: The **expectation** of probability density function f is

$$m = \int_{-\infty}^{\infty} x f(x) dx .$$

30.6. The second moment allows us to get the **variance** which is of great importance:

Definition: The **variance** of probability density function f is

$$\text{Var}(f) = \int_{-\infty}^{\infty} x^2 f(x) dx - m^2 ,$$

where m is the expectation. We can write $\text{Var}(f) = M_2 - M_1^2$.

Definition: The square root σ of the variance is called the **standard deviation**.

30.7. The standard deviation tells us what deviation we expect from the mean. Computing moments often leads to integration by parts problems:

Example: The expectation of the geometric distribution $f(x) = e^{-x}$

$$\int x e^{-x} dx = 1 .$$

Example: The variance of the geometric distribution $f(x) = e^{-x}$ is 1 and the standard deviation 1 too. To see this, let us compute

$$\int x^2 e^{-x} dx .$$

x^2	e^{-x}	
$2x$	$-e^{-x}$	\oplus
2	e^{-x}	\ominus
0	e^{-x}	\oplus

Example: You have already computed the expectation of the standard Normal distribution $f(x) = (2\pi)^{-1/2} e^{-x^2/2}$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-x^2/2} dx = 0 .$$

Example: The variance of the standard Normal distribution $f(x)$ is $\frac{1}{\sqrt{2\pi}}$ times

$$\int_{-\infty}^{\infty} x^2 e^{-x^2/2} dx .$$

We can compute this integral by partial integration too but we have to split it as $u = x$ and $v = x e^{-x^2/2}$.

$$-x e^{-x^2/2} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2} dx = \sqrt{2\pi} .$$

The variance therefore is $\boxed{1}$.

30.8. The next example is for trig substitution:

Example: The distribution supported on $[-1, 1]$ with function $(1/\pi)(1 - x^2)^{-1/2}$ there and 0 everywhere else is called the **arcsin-distribution**. What is the cumulative distribution function? What is the mean m ? What is the standard deviation σ ? We will compute this in class. The answers are $m = 0, \sigma = 1/\sqrt{2}$.

Homework

Problem 30.1: The function $f(x) = \cos(x)/2$ on $[-\pi/2, \pi/2]$ is a probability density function. Its mean is 0. Find its variance $\int_{-\pi/2}^{\pi/2} x^2 \cos(x) dx$.

Problem 30.2: The **uniform distribution on** $[a, b]$ is a distribution with probability density function is $f(x) = 1/(b - a)$ for $a \leq x \leq b$ and 0 elsewhere. Let $a = 1$ and $b = 5$;

a) Find the n 'th moment $M_n = \int_{-\infty}^{\infty} x^n f(x) dx$ in general.

b) Now compute the variance $\text{Var}[f] = M_2 - M_1^2$ and the standard deviation $\sigma = \sqrt{\text{Var}[f]}$.

Problem 30.3: Define $f(x)$ to be 0 for $x < 0$ and for $x > 0$ to be

$$f(x) = \frac{1}{\log(2)} \frac{e^{-x}}{1 + e^{-x}}.$$

a) Find the CDF F and verify that f is a probability density function. (Note that $F(x) = 0$ for $x \leq 0$ and especially $F(0) = 0$.)

b) Use a computer to numerically compute the expectation $m = M_1$.

c) Use a computer to compute the second moment M_2 .

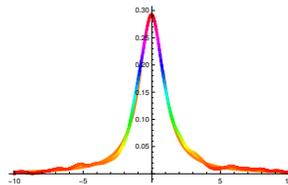
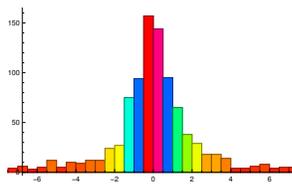
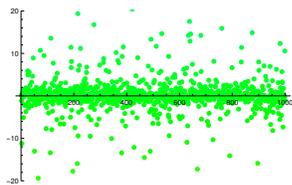
d) What is the standard deviation $\sigma = \sqrt{M_2 - M_1^2}$ of the distribution?

P.S. Your computer algebra system might tell you $M_1 = \zeta(2)/2$, $M_2 = 3\zeta(3)/2$. In general $M_n = \zeta(1 + n)n!(2^n - 1)/2^n$ for the **zeta function**.

Problem 30.4: a) Verify again that the **Cauchy distribution** with PDF $f(x) = \frac{(1/\pi)}{x^2+1}$ has the CDF $F(y) = 1/2 + \arctan(y)/\pi$.

b) What can you say about the variance of this distribution?

Problem 30.5: Let us quickly verify that if we take random numbers x in $[0, 1]$ then the data $\tan(x)$ are Cauchy distributed: just check that the probability that $y = \tan(\pi x)$ is in $[a, b]$ is $F(b) - F(a)$ for the Cauchy CDF appearing in the last problem. Now, use a calculator and compute 10 random Cauchy distributed numbers. In Mathematica such numbers can be accessed by `Tan[Pi * Random[]]`.



INTRODUCTION TO CALCULUS

MATH 1A

Unit 31: Calculus and Economics

LECTURE

31.1. Calculus is important in **economics**. This is an opportunity to review extrema problems and get acquainted with jargon in economics. Economists talk differently: $f' > 0$ means **growth** or **boom**, $f' < 0$ means **decline** or **recession**, a vertical asymptote is a **crash**, a horizontal asymptote a **stagnation**, a discontinuity is “**inelastic behavior**”, the derivative of something is the marginal of it: like marginal revenue.

Definition: The **marginal cost** is the derivative of the **total cost**.

31.2. Both, the marginal cost and total cost are functions of the quantity of goods produced.

Example: Assume the total cost function is $C(x) = 10x - 0.01x^2$. Find the marginal cost and the place where the total cost is minimal. **Solution.** Differentiate $C' = 10 + 0.02x$ Now find x which makes this vanish. We have $x = 50$.

Example: You sell spring water. The marginal cost to produce it is given by $f(x) = 10000 - x^4$. For which x is the total cost maximal?

Example: In the book "Dominik Heckner and Tobias Kretschmer: Don't worry about Micro, 2008", where the following strawberry story appears: (verbatim citation in italics):

Suppose you have all sizes of strawberries, from very large to very small. Each size of strawberry exists twice except for the smallest, of which you only have one. Let us also say that you line these strawberries up from very large to very small, then to very large again. You take one strawberry after another and place them on a scale that sells you the average weight of all strawberries. The first strawberry that you place in the bucket is very large, while every subsequent one will be smaller until you reach the smallest one. Because of the literal weight of the heavier ones, average weight is larger than marginal weight. Average weight still decreases, although less steeply than marginal weight. Once you reach the smallest strawberry, every subsequent strawberry will be larger which means that the rate of decrease of the average weight becomes smaller and smaller until eventually, it stands still. At this point the marginal weight is just equal to the average weight.

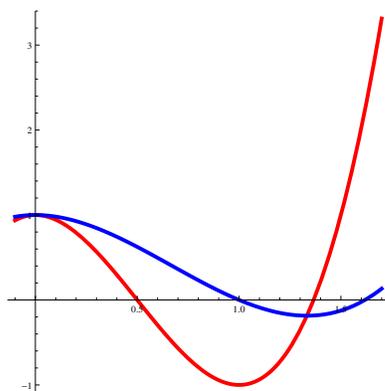


31.3. If $F(x)$ is the **total cost function** in dependence of the quantity x , then $F' = f$ is called the **marginal cost**.

Definition: The function $g(x) = F(x)/x$ is called the **average cost**.

Definition: A point where $f = g$ is called a **break-even point**.

Example: If $f(x) = 4x^3 - 3x^2 + 1$, then $F(x) = x^4 - x^3 + x$ and $g(x) = x^3 - x^2 + 1$. Find the break even point and the points where the average costs are extremal. **Solution:** To get the break even point, we solve $f - g = 0$. We get $f - g = x^2(3x - 4)$ and see that $x = 0$ and $x = 4/3$ are two break even points. The critical point of g are points where $g'(x) = 3x^2 - 4x$. They agree:



The following theorem tells that the marginal cost is equal to the average cost if and only if the average cost has a critical point. Since total costs are typically concave up, we usually have "break even points are minima for the average cost". Since the strawberry story illustrates it well, let's call it the "strawberry theorem":

Strawberry theorem: We have $g'(x) = 0$ if and only if $f = g$.

Proof.

$$g' = (F(x)/x)' = F'/x - F/x^2 = (1/x)(F' - F/x) = (1/x)(f - g).$$

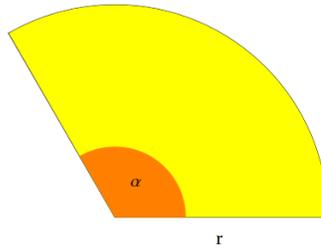
More extremization problems

31.4. In the second part of this lecture we still want to look at more extremization problems, also in the context of data.

Example: Find the rhomboid with side length 1 which has maximal area. Use an angle α to extremize.

Example: Find the sector of radius $r = 1$ and angle α which has minimal circumference $f = 2r + r\alpha$ if the area $r^2\alpha/2 = 1$ is fixed. **Solution.** Find $\alpha = 2/r^2$ from the second equation and plug it into the first equation. We get $f(r) = 2r + 2/r$. Now the task is to find the places where $f'(r) = 0$.

INTRODUCTION TO CALCULUS



Example: Find the ellipse of length $2a$ and width $2b$ which has fixed area $\pi ab = \pi$ and for which the sum of diameters $2a + 2b$ is maximal. **Solution.** Find $b = 1/a$ from the first equation and plug into the second equation. Again we have to extremize $f(a) = 2a + 2/a$. The same problem as before.

TO SEE HOW MARGINAL COST CURVES RELATE TO SUPPLY CURVES, LET'S LOOK AT ERNESTO'S COFFEE BUSINESS.

IT TURNS OUT THAT EVERY POINT ON ERNESTO'S SUPPLY CURVE...

MY SUPPLY CURVE SAYS THAT IF THE MARKET PRICE WERE \$2 PER CUP, ...

... I'D MAXIMIZE MY PROFIT BY SELLING 100 CUPS OF COFFEE PER HOUR.

... IS ALSO A POINT ON HIS MARGINAL COST CURVE!

THE MARGINAL COST OF PRODUCING THE 100TH CUP IS \$2.

THAT'S THE DIFFERENCE IN MY TOTAL COSTS BETWEEN PRODUCING 99 CUPS ...

... AND PRODUCING 100 CUPS!

THIS IS TRUE BECAUSE ERNESTO WANTS TO MAXIMIZE HIS PROFIT.

ERNESTO'S SUPPLY CURVE SAYS THAT IF THE MARKET PRICE WERE \$2 PER CUP, HE'D MAXIMIZE HIS PROFIT BY SELLING 100 CUPS.

BUT IF THE 100TH CUP COST MORE THAN \$2 TO PRODUCE...

... I COULD MAKE MORE PROFIT BY SELLING FEWER THAN 100 CUPS AT A MARKET PRICE OF \$2 PER CUP.

AND IF THE 100TH CUP COST LESS THAN \$2 TO PRODUCE...

... I COULD MAKE MORE PROFIT BY SELLING MORE THAN 100 CUPS AT A MARKET PRICE OF \$2 PER CUP.

SINCE HE'S PROFIT-MAXIMIZING, HIS COST OF PRODUCING THE 100TH CUP MUST BE \$2.

IF WE LOOK AT ERNESTO AND ALL THE OTHER COFFEE SELLERS TOGETHER, WE CAN SEE THAT EVERY POINT ON THE MARKET SUPPLY CURVE IS ALSO A POINT ON THE MARKET MARGINAL COST CURVE.

IF THE MARKET SUPPLY CURVE SAYS THAT AT A PRICE OF \$2 ALL THE SELLERS TOGETHER WANT TO SELL 20,000 CUPS OF COFFEE PER HOUR...

... THEN THE MARKET MARGINAL COST OF PRODUCING THE 20,000TH CUP MUST BE \$2.

AGAIN, THE REASON IS PROFIT MAXIMIZATION.

IF THE 20,000TH CUP COST MORE THAN \$2 TO PRODUCE...

... AT LEAST ONE OF US COULD MAKE MORE PROFIT BY SELLING FEWER CUPS AT A MARKET PRICE OF \$2!

AND IF THE 20,000TH CUP COST LESS THAN \$2 TO PRODUCE...

... AT LEAST ONE OF US COULD MAKE MORE PROFIT BY SELLING MORE CUPS AT A MARKET PRICE OF \$2!

ALL THESE LOGICAL ARGUMENTS CAN BE BACKED UP WITH ROCK-SOLID MATHEMATICS...

... BUT WE'D NEED TO DO SOME CALCULUS.

Facing market price p , a firm in a competitive market chooses quantity q to maximize profit π :

$$\pi = pq - c(q)$$

$$\frac{d\pi}{dq} = 0 \Rightarrow p = c'(q)$$

So either $q=0$ or the firm produces until marginal cost equals the market price!



Source: Grady Klein and Yoram Bauman, The Cartoon Introduction to Economics: Volume One Microeconomics, published by Hill and Wang. You can detect the strawberry theorem ($g' = 0$ is equivalent to $f = g$) can be seen on the blackboard.

Homework

Problem 31.1: a) Find the break-even point for an economic system if the marginal cost is $f(x) = 1/x$.
 b) Assume the marginal cost is $f(x) = x^7$. Verify that the average cost $g(x) = F(x)/x$ satisfies $8g(x) = f(x)$ and that $x = 0$ is the only break even point.

Problem 31.2: Let $f(x) = \cos(x)$. Compute $F(x)$ and $g(x)$ and verify that $f = g$ agrees with $g' = 0$.

Problem 31.3: The **production function** in an office gives the production $Q(L)$ in dependence of labor L . Assume

$$Q(L) = 5000L^3 - 3L^5 .$$

Find L which gives the maximal production.

This can be typical: For smaller groups, production usually increases when adding more workforce. After some point, bottle necks occur, not all resources can be used at the same time, management and bureaucracy is added, each person has less impact and feels less responsible, meetings slow down production etc. In this range, adding more people will decrease the productivity.



Problem 31.4: **Marginal revenue** f is the rate of change in **total revenue** F . As total and marginal cost, these are functions of the **cost** x . Assume the total revenue is $F(x) = -5x - x^5 + 9x^3$. Find the points, where the total revenue has a local maximum.

Problem 31.5: Find a line $y = mx$ through the points

$$(3, 4), (6, 3), (2, 5) ,$$

which minimize the function

$$f(m) = (3m - 4)^2 + (6m - 3)^2 + (2m - 5)^2 .$$

INTRODUCTION TO CALCULUS

MATH 1A

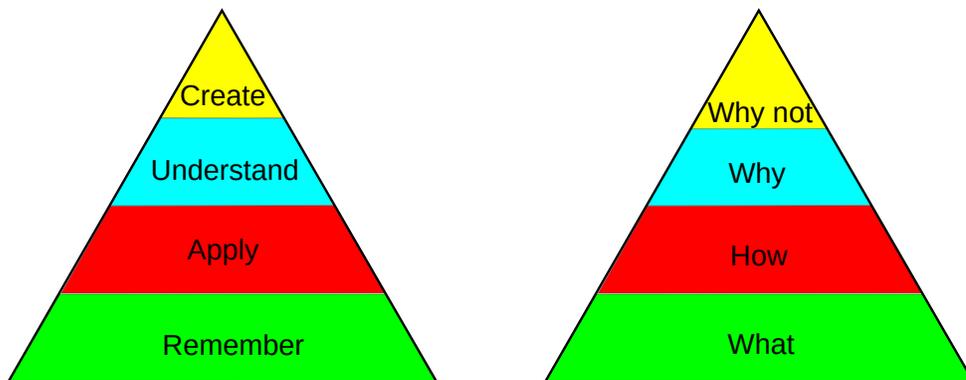
Unit 32: Calculus and AI

LECTURE

About intelligence

32.1. Artificial intelligence has a lot of overlap with education. If one wants to build an artificial intelligence entity, one needs to teach it first or to teach it how to learn itself. In a project of 2013, we have learned a bit about this by programming an AI bot called Sofia. It gave us quite a bit of insight on **how humans learn**. Teaching definitions for example is very simple. A machine is very good in memorizing stuff. Also teaching algorithms is no problem for a machine if the task is communicated clearly. What turns out to be much more difficult is to “teach insight”. How can one teach a machine for example to see what are “relevant” core principles, what is “important”, what is “good taste”. The last step is to teach being creative, or to discover new things. In that Sofia project, we boiled the process of learning and teaching down to the 4 questions “What, How, Why and Why not?”. We realized later that this turned out to be a variant of the **Bloom taxonomy** which splits learning into **Factual, Conceptual, Procedural** and **Metacognitive** parts.

32.2. Here is an example in calculus. We teach the concept of derivative. What is a derivative? How does one compute a derivative? Why does one compute derivatives? Why does one not just compute something else? It is no problem to teach a machine that the derivative of $f(x)$ is the limit $(f(x+h) - f(x))/h$ in the limit $h \rightarrow 0$. It is also no problem to teach the machine to take the derivative of a function like $\sin(5x)x$. All computer algebra systems know that already. The question why one wants to compute derivatives is harder. But it is teachable too. We want to compute derivatives for example because we want to find maxima or minima of quantities. The question of how we can extend the concept of derivative or replace it with something else is harder. There are other notions in mathematics which have done so. In quantum calculus, one looks at $Df(x) = f(x+1) - f(x)$ for example, in other parts of math, one has notions of “derivations” which formalize operations satisfying the Leibniz rule $(fg)' = f'g + fg'$. In quantum mechanics, the derivative is essentially the “momentum operator” generating translation. To teach a machine to come up with such connections and concepts is much, much harder.



A modified Bloom taxonomy. A major change is to put the “apply” part before the “understanding” part. The rationale is that in almost all situations of learning, one first learns how to do something before knowing why it works.

Generating calculus problems

32.3. We ask “Sofia”, our artificial intelligence teacher to automatically build worksheets or exam problems as well as solutions. In order to generate problems, we first must build **random functions**. When asked “give me an example of a function”, the system should generate functions of some complexity:

Definition: A **basic function** is a function from the 10 functions $\{\sin, \cos, \log, \exp, \tan, \text{sqt}, \text{pow}, \text{inv}, \text{sca}, \text{tra}\}$.

32.4. Here $\text{sqt}(x) = \sqrt{x}$ and $\text{inv}(x) = 1/x^k$ for a random integer k between -1 and -3 , $\text{pow}(x) = x^k$ for a random integer k between 2 and 5 . $\text{sca}(x) = kx$ is a scalar multiplication for a random nonzero integer k between -3 and 3 and $\text{tra}(x) = x + k$ translates for a random integer k between -4 and 4 .

32.5. Second, we use addition, subtraction multiplication, division and composition to build more complicated functions:

Definition: A **basic operation** is an operation from the list $\{f \circ g, f + g, f * g, f/g, f - g\}$.

32.6. The operation x^y is not included because it is equivalent to $\exp(x \log(y)) = \exp \circ (x \cdot \log)$. We can now build functions of various complexities:

Definition: A **random function** of complexity n is obtained by taking n random basic functions f_1, \dots, f_n , and n random basic operators $\oplus_1, \dots, \oplus_n$ and forming $f_n \oplus_n f_{n-1} \oplus_{n-1} \dots \oplus_2 f_1 \oplus_1 f_0$ where $f_0(x) = x$ and where we start forming the function from the right.

Example: Visitor: "Give me an easy function": Sofia looks for a function of complexity one: like $x \tan(x)$, or $x + \log(x)$, or $-3x^2$, or $x/(x - 3)$.

Example: Visitor: "Give me a function": Sofia returns a random function of complexity two: $x \sin(x) - \tan(x)$, or $-e^{\sqrt{x}} + \sqrt{x}$ or $x \sin(x)/\log(x)$ or $\tan(x)/x^4$.

Example: Visitor: "Give me a difficult function": Sofia builds a random function of complexity four like $x^4 e^{-\cos(x)} \cos(x) + \tan(x)$, or $x - \sqrt{x} - e^x + \log(x) + \cos(x)$, or $(1 + x)(x \cot(x) - \log(x))/x^2$, or $(-x + \sin(x + 3) - 3) \csc(x)$

32.7. Now, we can build a random calculus problem. To give you an idea, here are some templates for integration problems:

Definition: A **random integration problem** of complexity n is a sentence from the sentence list { "Integrate $f(x) = F(x)$ ", "Find the anti derivative of $F(x)$ ", "What is the integral of $f(x) = F(x)$?", "You know the derivative of a function is $f'(x) = F(x)$. Find $f(x)$." }, where F is a random function of complexity n .

Example: Visitor: "Give me a differentiation problem". **Sofia:** Differentiate $f(x) = x \sin(x) - \frac{1}{x^2}$. The answer is $\frac{2}{x^3} + \sin(x) + x \cos(x)$.

Example: Visitor: "Give me a difficult integration problem". **Sofia:** Find f if $f'(x) = \frac{1}{x} + (3 \sin^2(x) + \sin(\sin(x))) \cos(x)$. The answer is $\log(x) + \sin^3(x) - \cos(\sin(x))$.

Example: Visitor: "Give me an easy extremization problem". **Sofia:** Find the extrema of $f(x) = x/\log(x)$. The answer is $x = e$.

Example: Visitor: Give me an extremization problem". **Sofia:** Find the maxima and minima of $f(x) = x - x^4 + \log(x)$. The extrema are

$$\frac{\sqrt{(9 + \sqrt{3153})^{2/3} - 8\sqrt[3]{6}} + \sqrt{8\sqrt[3]{6} - (9 + \sqrt{3153})^{2/3} \left(1 + 6\sqrt{\frac{2}{9 + \sqrt{3153} - 8\sqrt[3]{6(9 + \sqrt{3153})}}}\right)}}{22^{5/6} \sqrt[3]{3} \sqrt[6]{9 + \sqrt{3153}}}$$

The last example shows the perils of random generation. Even so the function had decent complexity, the solution was difficult. Solutions can even be transcendental. This is not a big deal: just generate a new problem. By the way, all the above problems and solutions have been generated by Sofia. The dirty secret of calculus books is that there are maybe a thousand different type of questions which are usually asked. This is a reason why textbooks have become boring clones of each other and companies like "Aleks", "Demidec" etc exist which constantly mine the web and course sites like this and homework databases like "webwork" which contain thousands of pre-compiled problems in which randomness is already built in.

Automated problem generation is the "fast food" of teaching and usually not healthy. But like "fast food" has evolved, we can expect more and more computer assisting in calculus teaching.

Be assured that for this course, problems have been written by hand. (Sometimes Mathematica is used to see whether answers are reasonable). Handmade problems can sometimes a bit “rough” but can be more interesting. Still, you can see a worksheet which has been generated entirely by a program.

Homework

Problem 32.1: Lets build a differentiation problem by combining log and sin and exp. Differentiate all of the 6 combinations $\log(\sin(\exp(x)))$, $\log(\exp(\sin(x)))$, $\exp(\log(\sin(x)))$, $\exp(\sin(\log(x)))$, $\sin(\log(\exp(x)))$ and $\sin(\exp(\log(x)))$.

Problem 32.2: Four of the 6 combinations of log and sin and exp can be integrated as elementary functions.

- a) Find these cases
- b) Do these integrals.

Problem 32.3: From the 10 functions f and 10 functions g and 5 operations, we can build 500 functions. Some can not be integrated. An example is $\exp(\sin(x))$. Find 4 more which can not be integrated by you and computer algebra.

Problem 32.4: One of the most difficult things to teach is creativity. Lets try to be creative. Build an extremization problem which is applied. Here is an example (of course your example should be different than this):
A common theme for extrema are area and length. Invent an extremum problem involving an isoscele triangle. It should be of the form: "Maximize the area ... ". Now solve the problem.

Problem 32.5: Be creative: a) Create an area problem for a region in the plane which has not appeared in lecture, homework or exam.
 b) Create a volume problem for a surface of revolution. The problem should not have appeared yet in lecture, homework or exams. Make the problem so that it can be solved. Now solve your problem.

INTRODUCTION TO CALCULUS

MATH 1A

Lecture 33: Calculus and Music

Music is a function

33.1. Calculus is relevant in music because a music piece is just a **function**. If you feed a loudspeaker the function $f(t)$ the membrane gets displaced by $f(t)$. The pressure variations in the air are sound waves then reach your ear, where your ear drum oscillates as such. Plotting and playing works the same way. In Mathematica, we can play a function by replacing “Plot” with “Play” and say with

```
Play[ Sin[2Pi 1000 x^2], {x, 0, 10}]
```

33.2. While function f contains all the information about the music piece, the computer needs to store this as data. One possibility is in a “.WAV” file, which contains sampled values of the function usually with a sample rate of 44100 readings per second. Since our ear does not hear frequencies larger than 20'000 KHz, a sampling rate of 44.1K is good enough by a **theorem of Nyquist-Shannon**. More sophisticated storage possibilities exist. A .MP3 file for example encodes the function in a compressed way. To get from the sample values $f(k)$ the function back, the **Whittaker-Shannon interpolation formula**

$$f(t) = \sum_{k=1}^n f(k) \text{sinc}(t+k)$$

can be used. It involves the sinc function $\text{sinc}(x) = \sin(x)/x$ we have seen early in the course.

The wave form and hull

33.3. A periodic signal is the building block of sound. Assume $g(x)$ is a 2π -periodic function, we can generate a sound of 440 Hertz when playing the function $f(x) = g(440 \cdot 2\pi x)$. If the function does not have a smaller period, then we hear the **A tone**. It is a tone with 440 Hertz.

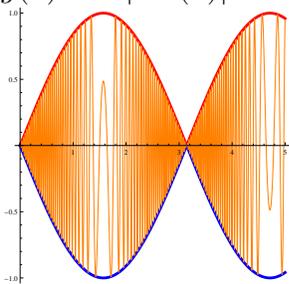
Definition: A periodic function g is called a **wave form**.



33.4. The wave form makes up the **timbre** of a sound which allows to model music instruments with macroscopic terms like **attack**, **vibrato**, **coloration**, **noise**, **echo**, **reverberation** and other characteristics.

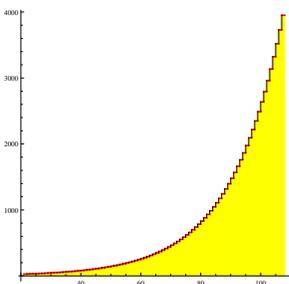
Definition: The **upper hull function** is defined as the interpolation of successive local maxima of f . The **lower hull function** is the interpolation of the local minima.

33.5. For the function $f(x) = \sin(100x)$ for example, the upper hull function is $g(x) = 1$ and the lower hull function is $g(x) = -1$. For $f(x) = \sin(x) \sin(100x)$ the upper hull function is approximately $g(x) = |\sin(x)|$ and the lower hull function is approximately $g(x) = -|\sin(x)|$.



We can not hear the actual function $f(x)$ because the function changes too fast that we can notice individual vibrations. But we can hear the hull function. We can hear **large scale amplitude** changes like **crescendi** or **diminuendi** or a **vibrato**. When playing two frequencies which are close, you hear **interference**.

The scale



33.6. Western music uses a discrete set of frequencies. This scale is based on the **exponential function**. The frequency f is an exponential function of the scale s . On the other hand, if the frequency is known then the scale number is a logarithm. This is a nice application of the logarithm:

Definition: A frequency f has the **Midi number** $s = 69 + 12 \cdot \log_2(f/440)$. The **piano scale function** or **midi function** gives back $f(s) = 440 \cdot 2^{(s-69)/12}$.

33.7. The Midi tone $s = 100$ for example gives $f = 2637.02$ Hertz.

The **piano scale function** $f(s) = 440 \cdot 2^{(s-69)/12}$ is an exponential function $f(s) = be^{as}$ which satisfies $f(s + 12) = 2f(s)$.

midifrequency [m.] := $\mathbf{N}[440 \cdot 2^{((m - 69)/12)}]$

33.8. The classical piano covers the 88 Midi tone scale from 21 to 108. It ranges from $f = 27.5Hz$, the sub-contra-octave A, to the highest $f = 4186.01Hz$, the 5-line octave C.

33.9. Filters: a function can be written as a sum of sin and cos functions. Our ear does this so called **Fourier decomposition** automatically. We can so hear melodies, filter out part of the music and hum it.

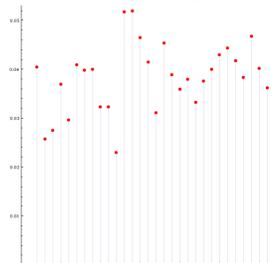
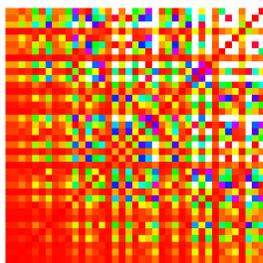
Pitch and autotune: it is possible to filter out frequencies and adapt their frequency. The popular filter **autotune** moves the frequencies around correcting wrong singing. If 440 Hertz (A) and 523.2 Hertz (C) for example were the only allowed frequencies, the filter would change a function $f(x) = \sin(2\pi 441x) + 4 \cos(2\pi 521x)$ to $g(x) = \sin(2\pi 440x) + 4 \cos(2\pi 523.2x)$. **Rip and remix:** if f and g are two songs, we can build the average $(f + g)/2$. A composer does this using **tracks**. Different instruments are recorded independently and then mixed together. A guitar $g(t)$, a voice $v(t)$ and a piano $p(t)$ together can form $f(t) = ag(t) + bv(t) + c(p(t))$ with suitably chosen constants a, b, c . **Reverberate and echo:** if f is a song and h is some time interval, we can look at $g(x) = Df(x) = [f(x+h) - f(x)]/h$. For small h , like $h = 1/1000$ the song does not change much because hearing $\sin(kx)$ or $\cos(kx)$ produces the same song. However, for larger h , one can get **reverberate** or **echo** effects.

Other math relations

33.10. Mathematics and music have a lot of overlap. Besides wave form analysis and music manipulation operations and symmetry, there are **encoding and compression problems**. A **Diophantine problem** is the question how well a frequency can be approximated by rationals. Why is the **chromatic scale** based on $2^{1/12}$ so effective? **Indian music** for example uses **micro-tones** and a scale of 22. The 12-tone scale has the property that many powers $2^{k/12}$ are close to rational numbers. This can be quantified with the **scale fitness**

$$M(n) = \sum_{k=1}^n \min_{p,q} |2^{k/n} - \frac{p}{q}| G(p, q)$$

where $G(n, m)$ is Euler's **gradus suavis** ("degree of pleasure") defined as $G(n, m) = 1 + E(nm/\text{gcd}(n, m))$ with **Euler gradus** $E(n) = \sum_{p|n} e(p)(p-1)$. The sum runs over all prime factors p of n and $e(p)$ is the multiplicity. The figure below shows that $n = 12$ has the best $M(n)$. The 2 could be replaced too. The **Stockhausen scale** uses $5^{k/25}$. You can hear it $f(t) = \sin(2\pi t 100 \cdot 5^{[t]/25})$, where $[t]$ is the largest integer smaller than t . The familiar **12-tone scale** can be admired by listening to $f(t) = \sin(2\pi t 100 \cdot 2^{[t]/12})$.



Example: The perfect fifth $3/2$ has the gradus suavis $1 + E(6) = 1 + 2 = 3$ which is the same than the perfect fourth $4/3$ for which $1 + E(12) = 1 + (2 - 1)(3 - 1)$. You can listen to the perfect fifth $f(x) = \sin(1000x) + \sin(1500x)$ or the perfect fourth $\sin(1000x) + \sin(1333x)$ and here is a function representing an **accord** with four notes $\sin(1000x) + \sin(1333x) + \sin(1500x) + \sin(2000x)$.

Homework

Problem 33.1: Modulation. Draw the hull function of the following functions.

a) $f(x) = \cos(200x) - \cos(201x)$

c) $f(x) = \sqrt{x} \cos(10000x)$

b) $f(x) = \cos(x) + \cos(\tan(1000\sqrt{x}))$

d) $f(x) = \cos(x) \sin(e^{2x})/2$

Here is how to play a function with Mathematica or Wolfram alpha:

```
Play[Cos[x] Sin[Exp[2 x]]/x, {x, 0,9}]
play sin(1000 x)
```

Problem 33.2: Amplitude modulation (AM): If you listen to $f(x) = \cos(x) \sin(1000x)$ you hear an amplitude change. Draw the hull function. How many increase in amplitudes to you hear in 10 seconds?

Problem 33.3: Other tonal scales, Midi number: As a creative musician, you create your own tonal scale. You decide to take the 8th root of 3 as your basic frequency change from one tone to the next.

a) After how many tonal steps has the frequency f tripled?

b) Build the midi function and its inverse for your tonal scale.

Problem 33.4: a) What is the frequency of the Midi number $s = 22$?

b) Which midi number belongs to the frequency $f = 2060 \text{ Herz}$?

Problem 33.5: Gradus Suavitatis. a) What is the gradus suavitatis of $25/64$? b) What is the gradus suavitatis of $5904/65536$?

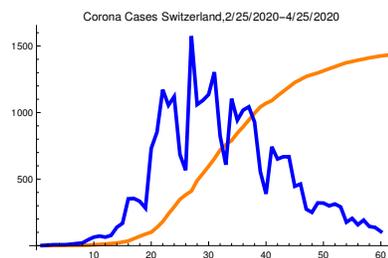
INTRODUCTION TO CALCULUS

MATH 1A

Unit 34: Data

DATA

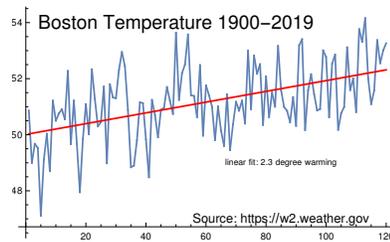
34.1. Information is stored in the form of **data**. Ultimately data can always be stored as numerical values $f(k)$, where k is a label. On a computer it is one function $f(1), \dots, f(n)$ with $f(k)$ taking values in $\{0, \dots, 254\}$, where k is the **memory address** and where n is the total number of **Bytes** it can store. Of course, these data are organized in a more convenient form like as a list of numbers, encoding a song or an array of numbers encoding a picture or an array of arrays of numbers encoding a movie. Here are the data of corona cases in Switzerland. They are given by an array $f(1) = 1, f(2) = 2, f(3) = 6, \dots, f(61) = 28611$ of numbers: ¹



34.2. Anything which has been developed in calculus can be applied to data. For example, we can look for differences, sum up data, average data, produce distributions. Knowing multi-variable calculus and linear algebra and probability theory helps a lot to model, visualize and reduce data. You should consider to learn multi-variable calculus and linear algebra in the future. It turns out however that most of the insight we can gain from data come from functions of one variable. We want to know the value $S(n)$ of a stock price, the temperature $T(n)$ or the number $I(n)$ of infected people by a disease on day n . Here is the development of the average year temperature in Boston over the last 120 years. ²

¹<https://www.corona-data.ch/>

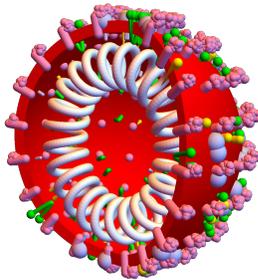
²<https://w2.weather.gov>



34.3. In some cases like illnesses, the data are estimates. For weather temperatures, they are quite accurate. A good model should allow to predict what the data should look like other times. But every model is just an application. A model can not only be either right or wrong, it can also be just irrelevant. Still, we can try build a model and then compare with the model. Both for pandemics as well as for climate change there are models. For climate we have a lot of accurate data, for pandemics, we do not.

POPULATION DYNAMICS

34.4. We are currently interested in the population dynamics of a virus “you know who”. Neil Shubin mentioned last week in a public virtual lecture that there are more viruses exist than stars in the universe, actually millions more ³ and that some of our genomes contain viral DNA so that we really owe our existence to the construct of viruses ⁴ This is a bit of a different perspective in a time we all lose our minds.



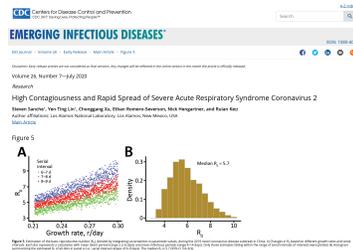
34.5. The simplest model for population dynamics is **exponential growth** given by the $f(t) = e^{ct}$ where c is a constant. In epidemiology, it is custom to define $f(t+h)/f(t) = R_0$, where R_0 is the **reproduction number** and h is an **infect period time**. This means with the notation introduced in the first week that $Df(t) = f(t+h) - f(t) = (R_0 - 1)f(t)$ so that $f(t) = (1+h)^{(R_0-1)t}f(0) = e^{ct}f(0)$ with $c = (R_0 - 1)\log(1+h)$. In the news, we often see R_0 values but not the **mean infection period** which is also important for estimating the growth. This spring one has tried through **increased hygiene** and **social distancing** the R_0 value down. You see in articles like in the figure that the R_0 value depends on the infection period h which is not yet precisely known. ⁵

³<https://www.airspacemag.com/daily-planet/there-are-more-viruses-earth-there-are-stars-universe-180974433>

⁴<https://www.nytimes.com/2017/10/04/science/ancient-viruses-dna-genome.html>

⁵Emerging infectious diseases, Volume 26, Number 7, July 2020 (early release)

INTRODUCTION TO CALCULUS



34.6. Once a certain saturation has been reached, the exponential growth slows down to a logistic growth. This happens as faster as larger the R_0 value is. The reason for the slow down is the decreasing number of new hosts assuming **immunity** of subjects which have overcome the disease. A logistic type growth is inevitable in a finite population and infection numbers of Corona are a textbook example for that. Especially politicians like to explain the slow-down also through social distancing. Others have pointed out that the R_0 value was actually smaller than now than before social distancing measures have come into effect and that simple measures like washing hands were already very effective. More radical ideas are trying to increase R_0 so that “herd immunity” kicks in earlier, minimizing the **overall harm** of a economic shut-down which also leads to life casualties, as other illnesses can no more be treated effectively, through stress etc. Lock-downs had also beneficial impacts like improved air quality due to less travel, good things for battling climate change.

POPULATION GROWTH

34.7. A follow up course to Math 1a is Math 1b, where one can learn more about series and differential equations. For example, one can look at the differential equation

$$f'(t) = rf(t)$$

which is an equation for an unknown quantity $f(t)$ like the number of infected people during the early stage of a pandemic. The rule tells that the rate of change (the increase of the number of infected people). It leads to **exponential growth**

$$f(t) = e^{rt}.$$

You can check that the derivative $f'(t)$ agrees with $rf(t)$.

34.8. If a population is finite, this exponential growth will slow down. The text book model is the **logistic growth** like $f'(t) = f(t)(1 - f(t))$ which is a **differential equation**. You can check that the function $f(t) = \frac{e^t}{e^t + 1}$ solves this equation. Can you do that? You have to check that the derivative $f'(t)$ agrees with $f(t)(1 - f(t))$.

35. LESSONS

35.1. The recent weeks have taught us various lessons. Here are few:

Data have to come with sources.

If a source does not do that and even government sites like CDC do not do that often, one has to complain. Unfortunately, most news outlets fail to do that. There should be a general rule that any statement comes with complete data sets.

35.2.

Accurate data are important for decisions.

35.3. If not enough data are available, one is flying blind and fear takes over. Fear can be a blessing as it leads to action and caution, it can also paralyze or lead to over-reactions.

Data can be interpreted in various ways.

35.4. There is a classical book by Darrell Huff from 1954 called “How to Lie with Statistics”. The recent months have given plenty of more examples.

It is possible to manipulate with pictures.

35.5. The book “**How to lie with statistics**” from the 1950ies which explains some manipulation tools. You have to be aware that manipulation is not always intended. Some authors want to justify a model, want to validate their data, want to influence people (for example to do trigger behavior in a crisis).

Also look critically statements of experts.

35.6. There are various forms of bias possible. People judge because of race, religion, gender, institution, title or appearance. We will look at examples like the **Dr Fox experiment**.

Look up the data yourself. How were the data obtained? How accurate are they?

35.7. Health data are difficult. On the CDC website, the annual burden of flue is the US estimated (April 26, 2020) with 12’000 to 61’000 deaths, 140’000-810’000 hospitalisations and 9.3 -45 million illnesses. Look at the large error margins. For example, hospitals are not always required to report influenza. With Covid 19, the problems have been discussed a lot. One does not know how many are infected, how many had it, nor how many deaths are really attributed to it as many victims had pre-conditions. So, unlike the temperature in Boston, the virus data can be interpreted in various ways.

35.8.

Even when highly complex multi-dimensional data exist, one dimensional visualizations are great.

So, even if you go beyond this course to other mathematical topics, single variable calculus will remain important!

INTRODUCTION TO CALCULUS

MATH 1A

Unit 35: Review

The older review sheets are unit 15 and unit 26.

Substitution

Substitution replaces $\int f(x) dx$ with $\int g(u) du$ with $u = u(x), du = u'(x)dx$. Cases:

- A) The integral of $f(x) = g(u(x))u'(x)$, is $G(u(x))$ where G is the anti derivative of g .
- B) $\int f(ax + b) dx = F(ax + b)/a$ where F is the anti derivative of f .

Examples:

A) $\int \sin(x^5)x^4 dx = \int \sin(u) du/5 = -\cos(u)/5 + C = -\cos(x^5)/5 + C$.

B) $\int \log(5x + 7) dx = \int \log(u) du/5 = (u \log(u) - u)/5 + C = (5x + 7) \log(5x + 7) - (5x + 7) + C$.

Integration by parts

A) Direct:

$$\int x \sin(x) dx = x (-\cos(x)) - \int 1 (-\cos(x)) dx = -x \cos(x) + \sin(x) + C dx .$$

B) Tic-Tac-Toe: to integrate $x^2 \sin(x)$

x^2	$\sin(x)$	
$2x$	$-\cos(x)$	\oplus
2	$-\sin(x)$	\ominus
0	$\cos(x)$	\oplus

The anti-derivative is

$$-x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) + C .$$

C) Merry go round: Example $I = \int \sin(x)e^x dx$. Use parts twice and solve for I .

Partial fractions

A) Make a common denominator on the right hand side $\frac{1}{(x-a)(x-b)} = \frac{A(x-b)+B(x-a)}{(x-a)(x-b)}$. and compare coefficients $1 = Ax - Ab + Bx - Ba$ to get $A + B = 0$, $Ab - Ba = 1$ and solve for A, B .

B) If $f(x) = p(x)/(x-a)(x-b)$ with different a, b , the coefficients A, B in $\frac{p(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$ can be obtained from

$$A = \lim_{x \rightarrow a} (x-a)f(x) = p(a)/(a-b), \quad B = \lim_{x \rightarrow b} (x-b)f(x) = p(b)/(b-a).$$

Examples:

A) $\int \frac{1}{(x+1)(x+2)} dx = \int \frac{A}{x+1} dx + \int \frac{B}{x+2} dx$. Find A, B by multiplying out and comparing coefficients in the nominator.

B) Directly write down $A = 1$ and $B = -1$, by plugging in $x = -2$ after multiplying with $x - 2$. or plugging in $x = -1$ after multiplying with $x - 1$.

Improper integrals

A) Integrate over infinite domain.

B) Integrate over singularity.

Examples:

A) $\int_0^{\infty} 1/(1+x^2) = \arctan(\infty) - \arctan(0) = \pi/2 - 0 = \pi/2$.

B) $\int_0^1 1/x^{2/3} dx = (3/1)x^{1/3}|_0^1 = 3$.

Trig substitution

When integrating function like $\sqrt{1-x^2}$, replace x by $\sin(u)$. **Example:**

$$\int_{-1}^1 \sqrt{1-x^2} dx = \int_{-\pi/2}^{\pi/2} \cos(u) \cos(u) du = \int_{-\pi/2}^{\pi/2} (1 + \cos(2u))/2 = \frac{\pi}{2}.$$

Terminology in Application

Music: hull function, piano function

Economics: average cost, marginal cost and total cost. Strawberry theorem

Operations research: find extrema, critical points, 2. derivative test

Computer science: random function from given function

Statistics: PDF, cumulative distribution function, expectation, variance.

Distributions: normal distribution, geometric distribution, Cauchy distribution.

Geometry: area between two curves, volume of solid

Numerical integration: Riemann sum, trapezoid rule, Simpson rule, Monte Carlo

Root finding: Bisection method, Newton method $T(x) = x - f(x)/f'(x)$.

Psychology: critical points and catastrophes, hate of related rate.

Physics: position, velocity and acceleration, work and power

Gastronomy: turn table to prevent wobbling, bottle calibration.

Checklists:

Integral techniques to consider

Try in the following order:

- Knowing the integral
- Substitution
- Trig substitution
- Integration by parts
- Partial fractions

Especially:

- Tic-Tac-Toe for integration by parts
- Hospital Method for partial fractions
- Merry go round method for parts

Integrals to know well

- $\sin(x)$
- $\cos(x)$
- $\tan(x)$
- $\log(x)$
- $\exp(x)$
- $1/x$
- $1/x^n$
- x^n
- \sqrt{x}
- $1/\cos^2(x)$
- $1/\sin^2(x)$
- $1/(1+x^2)$
- $1/(1-x^2)$
- $1/\sqrt{1-x^2}$
- $\sqrt{1-x^2}$

Applications you have to know

- Derivative:** Limit of differences $D_h f = [f(x+h) - f(x)]/h$ for $h \rightarrow 0$
- Integral:** Limit of Riemann sums $S_h f = [f(0) + f(h) + \dots + f(kh)]h$.
- Newton step:** $T(x) = x - f(x)/f'(x)$.
- Marginal cost:** the derivative F' of the total cost F .
- Average cost:** F/x where F is the total cost.
- Velocity:** Derivative of the position.
- Acceleration:** Derivative of the velocity.
- Curvature:** $f''(x)/(1 + f'(x)^2)^{3/2}$.
- Probability distribution function:** non-negative function with total $\int f(x)dx = 1$.
- Cumulative distribution function:** anti-derivative of the PDF.
- Expectation:** $\int xf(x) dx$, where f is the probability density function.
- Piano function:** frequencies $f(k) = 440 \cdot 2^{k/12}$ for integer k .
- Hull function:** $\sin(x) \sin(10000x)$ has hull $|\sin(x)|$
- Catastrophe:** A parameter c at which a local minimum disappears.

Core concepts

- Fundamental:** The fundamental theorem of calculus
- Extrema:** Second derivative test
- Derivatives:** slope rate of change
- Integrals:** area, volume
- Limits:** Hospital!
- Continuity:** know the enemies of continuity
- Numerics:** Riemann sum, Trapezoid and Simpson rule
- Rules:** Differentiation and integration rules.
- Methods:** Integration by parts, Substitution, Partial fraction.

Not needed on your fingertips

- Epidemic:** logistic growth.
- Entropy:** $-\int f(x) \log(f(x)) dx$.
- Monte Carlo integration:** $S_n = \frac{1}{n} \sum_{k=1}^n f(x_k)$, where x_k are random in $[a, b]$.
- Bart Simpson rule:** $S_n = \frac{1}{6n} \sum_{k=1}^n [f(x_k) + 4f(y_k) + f(x_{k+1})]$.
- Cocktail party stuff:** Eat, integrate and love, the story of exp in practice exam 2.
- Bottles:** How to calibrate bottles. The calibration formula.
- Sofia:** The name of a calculus bot once roaming the math department.
- Wobbly chair:** One can turn a chair on any lawn to stop it from wobbling.
- Song:** The hit: "low d high take high d low, cross the line and square the low"