INTRODUCTION TO CALCULUS

Unit 1: What is calculus?

LECTURE

1.1. Calculus deals with two things: **taking differences** and **summing things up**. Differences measure **change**, sums quantify how things **accumulate**. The process of taking differences has a limit called the **taking the derivative**. The process of taking sums has a limit called the **taking an integral**. These two operations are related. In this first lecture, we look at functions are evaluated on integers and where no limits are taken. It allows us to illustrate a major benefit of calculus: it gives us the ability to predict the future by analyzing the past.

1.2. Look at the following sequence of numbers

$$1, 7, 17, 31, 49, 71, ...$$

Can you figure out the next number in this sequence? When solving such riddles, we use already a basic idea of calculus. You might see that the differences

$$6, 10, 14, 18, 22, ...$$

already show a pattern. Taking differences again gives

$$4, 4, 4, 4, 4, ...$$

Now, we can go back to the previous sequence and see that that the next term is 26 and going to the original sequence gives $71 + 26 = 97$. Seeing the difference pattern allows us get the future terms in the sequence. This is important.

1.3. Let us rewrite what we just did using the concept of a function. A function f takes an input x and gives something out called $y = f(x)$. The sequence we have just seen can be seen as a function: $f(1) = 1, f(2) = 7, f(3) = 17, f(4) = 31, f(5) = 49, ...$ Define now a new function Df by $Df(x) = f(x+1) - f(x)$. It is a rate of change which we also call a "derivative". Write also $f'(x)$ instead of $f(x)$. We have $f'(1) = 6, f'(2) = 10, f'(3) = 14, ...$ Now, we can take the derivative again and define $f''(n) = f'(n+1) - f'(n)$. The function f'' is the function where the derivative has been applied twice. We have seen $f''(1) = 4, f''(2) = 4, f''(3) = 4, ...$ The second derivative is constant.

1.4. Functions can be visualized graphically in the form of a **graphs** $y = f(x)$. To do so, we draw two perpendicular axes, the x axes and the y axes and mark down every pair $(x, f(x))$ in that Euclidean plane.
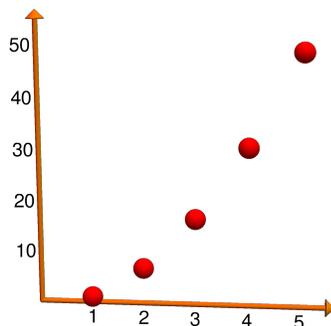
FIGURE 1. When plotting the sequence of numbers in the coordinate plane, we visualize the function as a graph.

1.5. When the first mathematicians were recording numbers they marked them into tally stick. An artifact from tens of thousands of years ago is the **Ishango bone**. We look at this as a constant function

$$1, 1, 1, 1, ... \ .$$

Over the next thousands of years, humans figured out to numbers using symbols like

$$1, 2, 3, 4, \dots \ .$$

We see that $1 = 1, 2 = 1 + 1, 3 = 1 + 1 + 1$ etc. If we look at this counting function $f(x) = x$, it satisfies $f'(x) = 1$, the constant function and $f''(x) = 0$. Which function g has the property that $g' = f$? It is the sum of the terms. For example $f(5) = 0 + 1 + 2 + 3 + 4$ and $f(3) = 0 + 1 + 2 + 3$, then $f(4+1) - f(4) = 4$. We see that if we define $g = Sf$ as

$$Sf(x) = f(0) + f(1) + f(2) + \dots + f(x - 1)$$

then $g(x + 1) - g(x) = f(x)$. Can we get a formula for that?

1.6. The new function g satisfies $g(1) = 1, g(2) = 3, g(3) = 6$, etc. These numbers are called **triangular numbers**. From the function g we can get f back by taking difference:

$$Dg(n) = g(n + 1) - g(n) = f(n) \ .$$

For example $Dg(5) = g(6) - g(5) = 15 - 10 = 5$. And indeed this is $f(5)$. Finding a formula for the sum $Sf(n)$ is not so easy. We have to find the n'th term in the sequence which starts with

$$1, 3, 6, 10, 15, 21, \dots$$

1.7. Legend tells that when **Karl-Friedrich Gauss** was a 9 year old school kid, his teacher, Mr. Büttner gave him the task to sum up the first 100 positive integers $1 + 2 + \dots + 100$. Gauss did not want to do this tedious work and looked for a better way to do it. He discovered that pairing the numbers up would simplify the summation He would write the sum as $(1 + 100) + (2 + 99) + \dots + (50 + 51)$ so that the answer is $g(x) = x(x - 1)/2 = 5050$. We have now an explicit expression for the sum function. Lets apply the difference function again: $Dg(x) = x(x + 1)/2 - x(x - 1)/2 = x = f(x)$.

1.8. Let us add up the new sequence again and compute $h = Sg$. We get the sequence $0, 1, 4, 10, 20, 35, \ldots$ called **tetrahedral numbers**. The reason is that one can use $h(n)$ balls to build a tetrahedron of side length n. For example, we need $h(4) = 20$ golf balls to build a tetrahedron of side length 4. The formula which holds for h is $\boxed{h(x) = x(x-1)(x-2)/6}$. In the worksheet we will check that summing the differences gives the function back.

1.9. The general relation
$$\boxed{SDf(x) = f(x) - f(0), \qquad DSf(x) = f(n)}$$
is an arithmetic version of the **fundamental theorem of calculus**. It will lead to the **integral** $\boxed{\int_0^x f(x)\,dx}$. **derivative** $\boxed{\frac{d}{dx}f(x)}$ and the **fundamental theorem of calculus**
$$\boxed{\int_0^x \frac{d}{dt}f(t)\,dt = f(x) - f(0), \qquad \frac{d}{dx}\int_0^x f(t)\,dt = f(x)}$$

1.10. It is a fantastic result. The goal of this course is to understand this theorem and apply it. [1] The above version will lead us. Note that if we define $[n]^0 = 1, [n]^1 = n, [n]^2 = n(n-1)/2, [n]^3 = n(n-1)(n-2)/6$ then $D[n] = [1], D[n]^2 = 2[n], D[n]^3 = 3[n]^2$ and in general
$$\boxed{\frac{d}{dx}[x]^n = n[x]^{n-1}}$$

<div align="center">EXAMPLES</div>

1.11. The Fibonnacci sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ satisfies $f(x) = f(x-1) + f(x-2)$. It defines a function on the positive integers. For example, $f(6) = 8$. What is the function $g = Df$? We can assume $f(0) = 0$. **Solution:** We take the difference between successive numbers and get the same sequence again but shifted. We have $\boxed{Df(x) = f(x-1)}$.

1.12. Take the same function f given by the sequence $1, 1, 2, 3, 5, 8, 13, 21, \ldots$ but now compute the function $h(n) = Sf(n)$ obtained by summing the first n numbers up. It gives the sequence $1, 2, 4, 7, 12, 20, 33, \ldots$. What sequence is that?
Solution: Because $Df(x) = f(x-1)$ we have $f(x) - f(0) = SDf(x) = Sf(x-1)$ so that $Sf(x) = f(x+1) - f(1)$. Summing the Fibonnacci sequence produces the Fibonnacci sequence shifted to the left with $f(2) = 1$ is subtracted.

1.13. Find the next term in the sequence $2 \quad 6 \quad 12 \quad 20 \quad 30 \quad 42 \quad 56 \quad 72 \quad 90 \quad 110 \quad 132$.
Solution: Take differences

$$
\begin{array}{ccccccccccc}
2 & 6 & 12 & 20 & 30 & 42 & 56 & 72 & 90 & 110 & 132 \\
2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{array} \ .
$$

Now we can add an additional number, starting from the bottom and working us up.

$$
\begin{array}{cccccccccccc}
2 & 6 & 12 & 20 & 30 & 42 & 56 & 72 & 90 & 110 & 132 & \boxed{156} \\
2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & 22 & \boxed{24} \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & \boxed{2}
\end{array}
$$

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[1]Many textbooks need hundreds of pages until the fundamental theorem is reached.

1.14. Look at the function $f(n)$ which gives the n'th prime number. Lets look at the derivatives $D^k f$ but take the absolute value $|D^k(f)|$. In other words, we study $T(f)(n) = |f(n+1) - f(n)|$. We know for example that $f(n) = 2^n$ satisfies $Tf = f$. Lets look at the prime function and apply this differences:

n=	1	2	3	4	5	6	7	8	9	...
$f(n) =$	2	3	5	7	11	13	17	23	29	...
$Tf(n) =$	1	2	2	4	2	4	2	4	6	...
$T^2 f(n) =$	1	0	2	2	2	2	2	2	4	...
$T^3 f(n) =$	1	2	0	0	0	0	0	2	0	...

The **Gilbreath conjecture** of 1959 claims that the first entry remains 1 for ever when applying this absolute differentiation process. The problem is still open.

HOMEWORK

This homework is due the next class

Problem 1.1: Predict the future and find the next term in the sequence

$$2, 10, 30, 68, 130, 222, 350, 520, 738, 1010, 1342, ...$$

by taking "derivatives" and then "integrating".

Problem 1.2: Look at the odd numbers $f(n) = 2n + 1$. The sequence starts with $1, 3, 5, 7, 9, 11, 13, \ldots$. We want to find a function g which has the property $Dg = f$. To do so, play around and compute $Sf(1) = 1, Sf(2) = 1 + 3, Sf(3) = 1 + 3 + 5$ etc until you see a pattern. Now guess a formula for $g(n) = Sf(x) = f(0) + f(1) + f(2) + \cdots + f(x-1)$ and verify algebraically that $Dg(x) = g(x+1) - g(x) = f(x)$ indeed holds.

Problem 1.3: The function $f(x) = 2^x$ is called the **exponential function**. We have for example $f(0) = 1, f(1) = 2, f(2) = 4, \ldots$. Verify that this function satisfies the equation $\boxed{Df(x) = f(x)}$. The derivative of the exponential function is the exponential function itself.

Problem 1.4: It is believed that the prime function $f(x)$ which gives the x'th prime has infinitely many values, where $f'(x) = 2$. These are called **prime twins**. Find at least 10 points x, where the "derivative" of the prime function is indeed $f'(x) = 2$.

Problem 1.5: Check in each of the following two cases that $g = Sf$ holds. You can do that by verifying that $Dg = f$.
a) For $f(x) = x + 1$ we have $g(x) = x(x+1)/2$.
b) For $f(x) = (x-1)(x-2)/2$ we have $g(x) = x(x-1)(x-2)/6$.