

INTRODUCTION TO CALCULUS

MATH 1A

Unit 8: Derivative Function

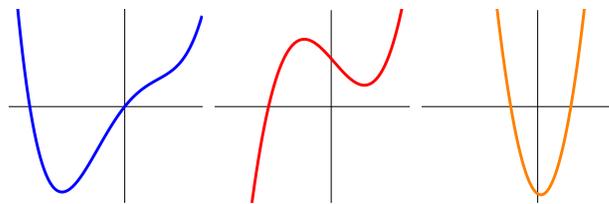
LECTURE

8.1. We have defined the derivative $f'(x) = \frac{d}{dx}f(x)$ as a limit of $(f(x+h) - f(x))/h$ for $h \rightarrow 0$. We have seen that $\frac{d}{dx}x^n = nx^{n-1}$ holds for integer n . We also know already that $\sin'(ax) = a \cos(x)$, $\cos'(ax) = -a \sin(x)$ and $\exp'(ax) = a \exp(ax)$. We can already differentiate a lot of functions and evaluate the derivative $f'(x)$ at some point x and so compute the slope of the curve at x .

8.2. Example: Find the derivative $f'(x)$ of $f(x) = \sin(4x) + \cos(5x) - \sqrt{x} + 1/x + x^4$ and evaluate it at $x = 1$. **Solution:** $f'(x) = 4 \cos(4x) - 5 \sin(5x) - 1/(2\sqrt{x}) - 1/x^2 + 4x^3$. Plugging in $x = 1$ gives $-\pi - 1/2 - 1 + 4$.

8.3. One can look at the differentiation process also as a rule which assigns to a function f a new function f' , the **derivative function**. For example, for $f(x) = \sin(x)$, we get $f'(x) = \cos(x)$. In this lecture, we want to understand the new function and its relation with f . What does it mean if $f'(x) > 0$. What does it mean that $f'(x) < 0$. Do the roots of f tell something about f' or do the roots of f' tell something about f ?

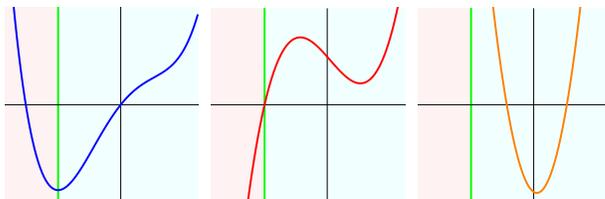
8.4. Here is an example of a function f , its derivative f' and the derivative of the derivative f'' . Can you see the relation?



8.5. To understand the relation, it is good to distinguish intervals, where $f(x)$ is increasing or decreasing. These are the intervals where $f'(x)$ is positive or negative.

Definition: A function is called **monotonically increasing** on an interval $I = (a, b)$ if $f'(x) > 0$ for all $x \in (a, b)$. It is **monotonically decreasing** if $f'(x) < 0$ for all $x \in (a, b)$.

Monotonically increasing functions “go up” when you “increase x”. Let's color that:



Example: Can you find a function f on the interval $[0, 1]$ which is bounded $|f(x)| \leq 1$ but such that $f'(x)$ is unbounded? Hint: square-”...”-beer.

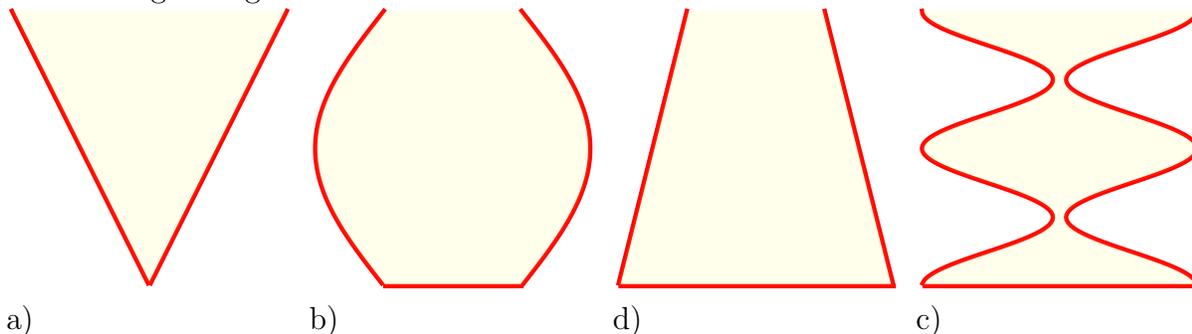
Definition: Given the function $f(x)$, we can define $g(x) = f'(x)$ and then take the derivative g' of g . This second derivative $f''(x)$ is called the **acceleration**. It measures the rate of change of the tangent slope. For $f(x) = x^4$, for example we have $f''(x) = 12x^2$. If $f''(x) > 0$ on some interval the function is called **concave up**, if $f''(x) < 0$, it is **concave down**.

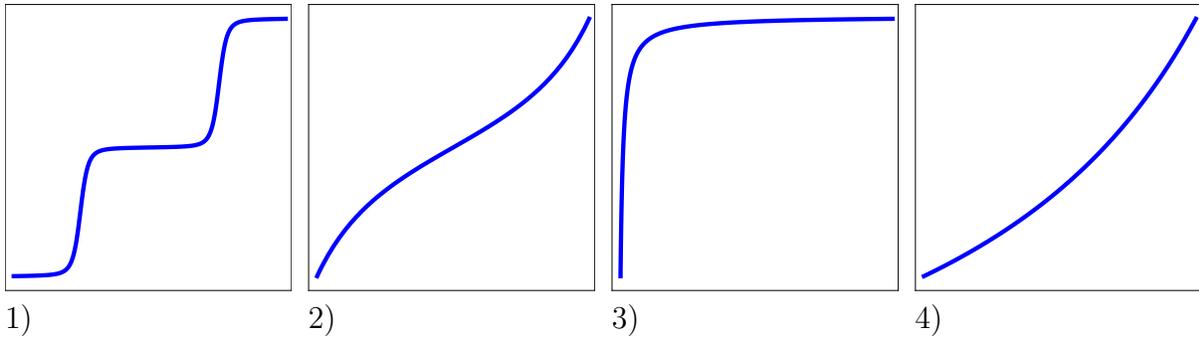
Example: Find a function f which has the property that its acceleration is constant equal to 6. **Solution.** We have to get a function such that its derivative is $6x$. That works for $3x^2$.

Example: Find a function f which has the property that its acceleration is equal to its derivative. To do so, try basic functions you know and compute $f'(x), f''(x)$ in each case.

8.6. In the famous **bottle calibration problem**, we fill a circular bottle or glass with constant amount of fluid. Plot the height of the fluid in the bottle at time t . Assume the radius of the bottle is $f(z)$ at height z . Can you find a formula for the height $g(t)$ of the water? This is not so easy. But we can find the rate of change $g'(t)$. Assume for example that f is constant, then the rate of change is constant and the height of the water increases linearly like $g(t) = t$. If the bottle gets wider, then the height of the water increases slower. There is definitely a relation between the rate of change of g and f . Before we look at this more closely, lets try to match the following cases of bottles with the graphs of the functions g qualitatively.

Example: In each of the bottles, we call g the height of the water level at time t , when filling the bottle with a constant stream of water. Can you match each bottle with the right height function?





8.7. The key is to look at $g'(t)$, the rate of change of the height function. Because $[g(t+h) - g(t)]$ times the area πf^2 is a constant times the time difference $h = dt$, we have **bottle calibration formula**

$$g' = \frac{1}{\pi f^2}.$$

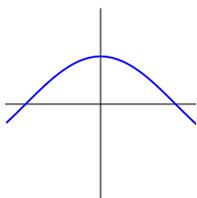
It relates the derivative function of g with the thickness $f(t)$ of the bottle at height g . No need to learn this. It just explains the story completely. It tells that that if the bottle radius f is large, then the water level increase g' is small and if the bottle radius f is small, then the liquid level change g' is large.

Homework

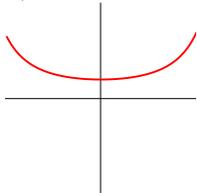
Problem 8.1: For the following functions, determine on which intervals the function is monotonically increasing or monotonically decreasing.

- a) $f(x) = -x^3 + x$ on $[-2, 2]$
- b) $f(x) = \sin(x)$ on $[-2\pi, 2\pi]$
- c) $f(x) = -x^4 + 8x^2$ on $[-4, 4]$.

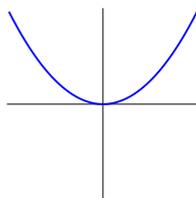
Problem 8.2: Match the following functions with their derivatives. Explain using monotonicity



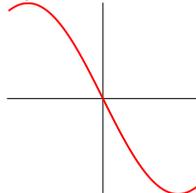
a)



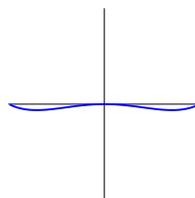
1)



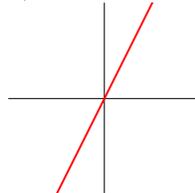
b)



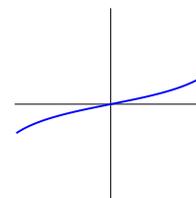
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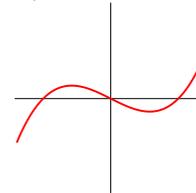
c)



3)

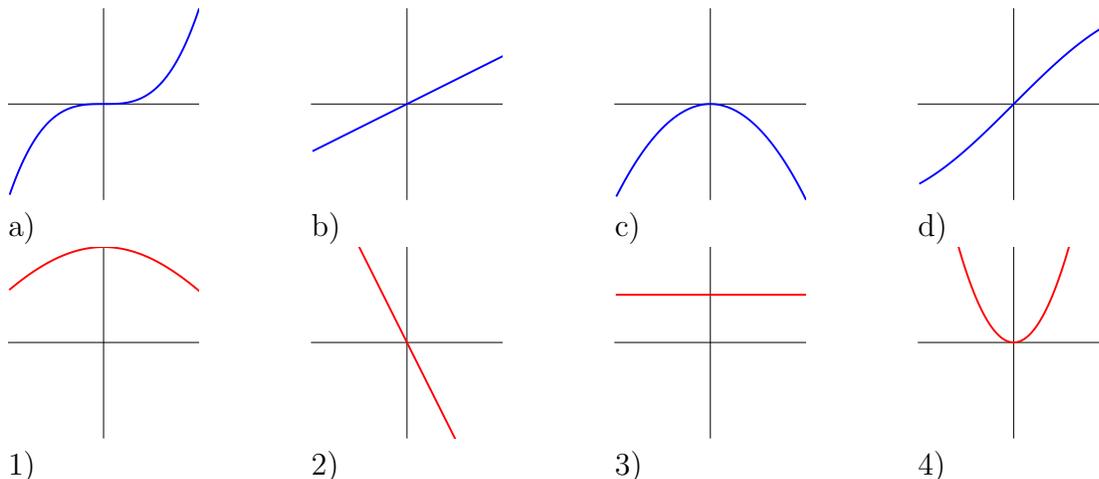


d)



4)

Problem 8.3: Match also the following functions with their derivatives. Give short explanations documenting your reasoning in each case.



Problem 8.4: Draw for the following functions the graph of the function $f(x)$ as well as the graph of its derivative $f'(x)$. You do not have to compute the derivative analytically as a formula here since we do not have all tools yet to compute the derivatives. The derivative function you draw needs to have the right qualitative shape however.

a) The **”To whom the bell tolls”** function

$$f(x) = e^{-x^2}$$

b) The **”Maria Agnesi”** function:

$$f(x) = \frac{1}{1+x^2}$$

c) The **three gorges** function

$$f(x) = \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1} .$$

Problem 8.5: Below you the graphs of three derivative functions $f'(x)$. In each case you are told that $f(0) = 1$. Your task is to draw the function $f(x)$ in each of the cases a),b),c),d). Your picture does not have to be up to scale, but your drawing should display the right features.

