INTRODUCTION TO CALCULUS

MATH 1A

Unit 9: Product Rule

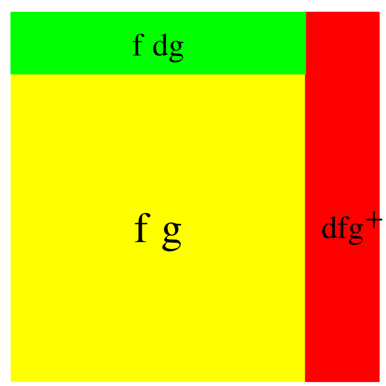
Lecture

9.1. In this lecture, we look at the derivative of a product of functions. The product rule is also called **Leibniz rule** named after **Gottfried Leibniz**, who found it in 1684. It is important because it allows us to differentiate many more functions. We will be able to compute so the derivative of $f(x) = x \sin(x)$ for example without having to take the limit the limit $\lim(f(x+h) - f(x))/h$. Let us start with an identity without limits. It is a discrete Leibniz rule which holds in the **Babylonian calculus** developed in the first hour.

$$f(x+h)g(x+h) - f(x)g(x) = [f(x+h) - f(x)] \cdot g(x+h) + f(x) \cdot [g(x+h) - g(x)] \; .$$

9.2. It can be written as $D(fg) = Dfg^+ + fDg$ with a shifted function $g^+(x) = g(x+h)$ and $Df(x) = [f(x+h) - f(x)]/h$. This **quantum Leibniz rule** can also be seen geometrically: the rectangle of area $(f+df)(g+dg)$ is the union of rectangles with area $f \cdot g$, $f \cdot dg$ and $df \cdot g^+$. Now take the limit $h \to 0$:

$$\frac{[f(x+h)-f(x)]}{h} \cdot g(x+h) \to f'(x) \cdot g(x)$$
$$f(x) \cdot \frac{[g(x+h)-g(x)]}{h} \to f(x) \cdot g'(x)$$



9.3. We get the extraordinarily important **product rule**:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x) \; .$$

Example: Find the derivative function $f'(x)$ for $f(x) = x^3 \sin(x)$. **Solution:** We know how to differentiate x^3 and $\sin(x)$ so that $f'(x) = 3x^2 \sin(x) + x^3 \cos(x)$.

Example: While we know for $f(x) = x^5$ that $f'(x) = 5x^4$, let us compute this with the Leibniz rule. To do so, write $x^5 = x^4 \cdot x$. We have

$$\frac{d}{dx}x^4 = 4x^3, \frac{d}{dx}x = 1 .$$

The product rule gives us $f'(x) = 4x^3 \cdot x + x^4 \cdot 1 = 5x^4$. In principle we could be induction prove so the formula $f'(x^n) = nx^{n-1}$.

Example: We look now at a few derivatives related to functions, where we know the answer already but where we can check things using the product formula:

- $\frac{d}{dx}(x^3 \cdot x^5)$
- $\frac{d}{dx}e^{3x}e^{5x}$
- $\frac{d}{dx}\sqrt{x}/\sqrt{x}$
- $\frac{d}{dx}\sin(x)\cos(x)$

9.4. There is also a quotient rule which allows to differentiate $f(x)/g(x)$. Because we can write this as $f(x) \cdot 1/g(x)$, we only need to know how to differentiate $1/g(x)$. This is the **reciprocal rule**:

If $g(x) \neq 0$, then

$$\frac{d}{dx}\frac{1}{g(x)} = \frac{-g'(x)}{g(x)^2} .$$

9.5. In order to see this, write $h = 1/g$ and differentiate the equation $1 = g(x)h(x)$ on both sides. The product rule gives $0 = g'(x)h(x) + g(x)h'(x)$ so that $h'(x) = -h(x)g'(x)/g(x) = -g'(x)/g^2(x)$.

Example: Find the derivative of $f(x) = 1/x^4$. **Solution**: $f'(x) = -4x^3/x^8 = -4/x^5$. The same computation shows that $\frac{d}{dx}x^n = nx^{n-1}$ holds for all integers n.

The formula $\boxed{\frac{d}{dx}x^n = nx^{n-1}}$ holds for all integers n.

9.6. The **quotient rule** is obtained by applying the product rule to $f(x) \cdot (1/g(x))$ and using the reciprocal rule. This gives the **"Low D High take High D Low - cross the line and square the Low"** rule:

If $g(x) \neq 0$, then

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{[g(x)f'(x) - f(x)g'(x)]}{g^2(x)} .$$

Example: Find the derivative of $f(x) = \tan(x)$. **Solution**: because $\tan(x) = \sin(x)/\cos(x)$ we have

$$\tan'(x) = \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} .$$

Example: Find the derivative of $f(x) = \frac{2-x}{x^2+x^4+1}$. **Solution**. We apply the quotient rule and get $[(-1)x^2 + x^4 + 1 + (2-x)(2x + 4x^3)]/(x^2 + x^4 + 1)$.

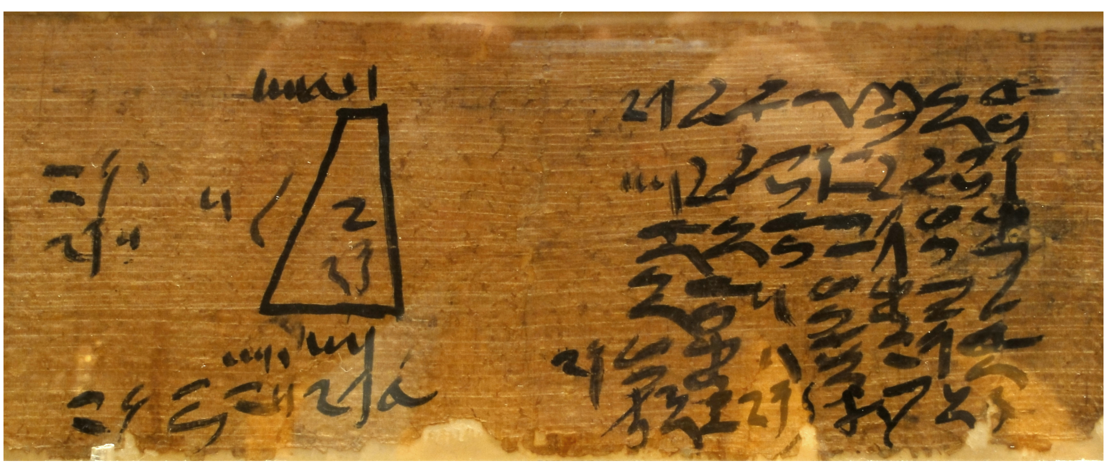
9.7. Here are some more problems with solutions:

Example: Find the second derivative of $\tan(x)$. **Solution**. We have already computed $\tan'(x) = 1/\cos^2(x)$. Differentiate this again with the quotient rule gives $\frac{-\frac{d}{dx}\cos^2(x)}{\cos^4(x)}$. We still have to find the derivative of $\cos^2(x)$. The product rule gives $\cos(x)\sin(x) + \sin(x)\cos(x) = 2\cos(x)\sin(x)$. This gives $2\sin(x)/\cos^3(x)$.

Example: A cylinder has volume $V = \pi r^2 h$, where r is the radius and h is the height. Assume the radius grows like $r(t) = 1 + t$ and the height shrinks like $1 - \sin(t)$. Does the volume grow or decrease at $t = 0$?

Solution: The volume $V(t) = \pi(1+t)^2(1 - \sin(t))$ is a product of two functions $f(t) = \pi(1+t)^2$ and $g(t) = (1-\sin(t)$. We have $f(0) = 1, g'(0) = 2, f'(0) = 2, g(0) = 1$. The product rule gives gives $V'(0) = \pi 1 \cdot (-1) + \pi 2 \cdot 1 = \pi$. The volume increases in volume at first.

9.8. On the **Moscow papyrus** dating back to 1850 BC, the general formula $V = h(a^2 + ab + b^2)/3$ for a truncated pyramid with base length a, roof length b and height h appeared. Assume $h(t) = 1+\sin(t), a(t) = 1+t, b(t) = 1-2t$. Does the volume of the truncated pyramid grow or decrease at first? **Solution**. We could fill in $a(t), b(t), h(t)$ into the formula for V and compute the derivative using the product rule. A bit faster is to write $f(t) = a^2 + ab + b^2 = (1+t)^2 + (1-3t)^2 + (1+t)(1-3t)$ and note $f(0) = 3, f'(0) = -6$ then get from $h(t) = (1+\sin(t))$ the data $h(0) = 1, h'(0) = 1$. So that $V'(0) = (h'(0)f(0) - h(0)f'(0))/3 = (1 \cdot 3 - 1(-6))/3 = -1$. The pyramid shrinks in volume at first.



Example: We pump up a balloon and let it fly. Assume that the thrust increases like t and the resistance decreases like $1/\sqrt{1-t}$ since the balloon gets smaller. The distance traveled is $f(t) = t/\sqrt{1-t}$. Find the velocity $f'(t)$ at time $t = 0$.

Homework

Problem 9.1: Find the derivatives of the following functions, then evaluate at $x = 0$
a) $f(x) = \sin(3x) + \sin(120x)\tan(121x)$.
b) $f(x) = \cos^4(x)/(1+x)^5$.
c) $f(x) = e^x \sin(x)\cos(x)$.
d) $f(x) = 3/\cos(x) + 1/\sqrt{x+1}$.
e) $f(x) = 6xe^{5x} + 8\tan(x)$.

Problem 9.2: a) Verify that for $f(x) = g(x)h(x)k(x)l(x)$ the formula $f' = g'hkl + gh'kl + ghk'l + ghkl'$ holds.
b) Verify the following formula for derivative of $f(x) = g(x)^4$ we have $f'(x) = 4g^3(x)g'(x)$. Do not use the chain rule but $f(x) = g(x)g(x)g(x)g(x)$ and a).

Problem 9.3: If $f(x) = \text{sinc}(x) = \sin(x)/x$, find its derivative $g(x) = f'(x)$ and then the derivative of $g(x)$. Then evaluate this at $x = 0$.

Problem 9.4: a) Find the derivative of

$$\frac{\sin(x)}{1 + \cos(x) + \frac{x^4}{\sin(x)}}$$

at $x = 0$. Try to do this as effectively as you can.
b) No gain without pain: find the derivative of

$$f(x) = \frac{1}{\sin(x) + \frac{1}{\sin(x)+1/(\sin(x)+1/\sin(x))}}$$

at $x = \pi/2$.

Problem 9.5: Here is a preparation for the chain rule, we see in the next unit. But please avoid the chain rule in a) and b) .
a) We have already computed the derivative of $g(x) = \sqrt{x}$ in the last homework. Introduce $f(x) = x^{1/4}$ and apply the product rule to $g(x) = f(x)f(x)$ to get the derivative of f.
b) Use problem 2b) applied to the identity $x = f(x)^4$ to get the derivative of f.
c) Now remember the chain rule and use it to get $f'(x)$. If you have not seen the chain rule, no problem, just look it up. We will cover it next time.