INTRODUCTION TO CALCULUS

MATH 1A

Unit 14: Newton method

Lecture

14.1. In the lecture on the intermediate value theorem, we found roots of functions using a "divide and conquer" technique: start with an interval $[a, b]$ for which $f(a) < 0$ and $f(b) > 0$. If $f((a + b)/2)$ is positive, then use the interval $[a, (a + b)/2]$ otherwise $[(a + b)/2, b]$. After n steps, we are $(b - a)/2^n$ close to the root. If the function f is differentiable, we can do better and use the value of the derivative to get closer to a point $y = T(x)$. Lets find this point y. If we draw a tangent at $(x, f(x))$ and intersect it with the x-axes, then

$$f'(x) = \frac{f(x) - 0}{x - T(x)} \ .$$

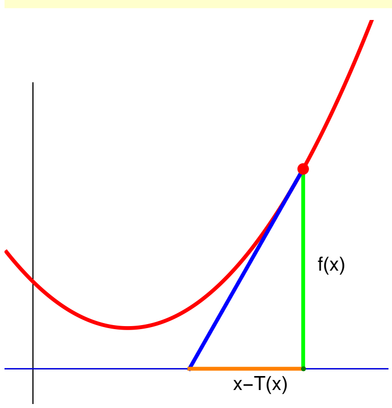
Now, $f'(x)$ is the slope of the tangent and the right hand side is "rise" over "run" (see the picture). If we solve for $T(x)$, we get

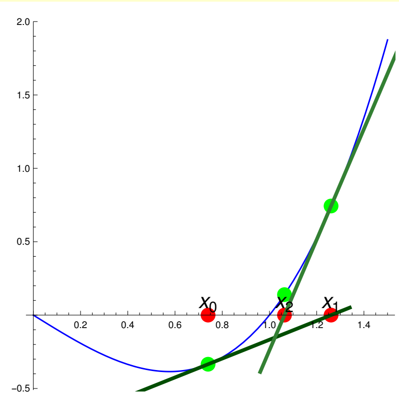
Definition: The **Newton map** is defined as

$$T(x) = x - \frac{f(x)}{f'(x)} \ .$$

14.2. The **Newton's method** applies this map a couple of times until we are sufficiently close to the root: start with a point x, then compute a new point $x_1 = T(x)$, then $x_2 = T(x_1)$ etc.

If p is a root of f such that $f'(p) \neq 0$, and x_0 is close enough to p, then $x_1 = T(x), x_2 = T^2(x)$ converges to the root p.





Example: If $f(x) = ax + b$, we reach the root in one step.

Example: If $f(x) = x^2$ then $T(x) = x - x^2/(2x) = x/2$. We get exponentially fast to the root 0. In general, the method is much better:

The Newton method converges extremely fast to a root $f(p) = 0$ if $f'(p) \neq 0$. In general, the number of correct digits double in each step.

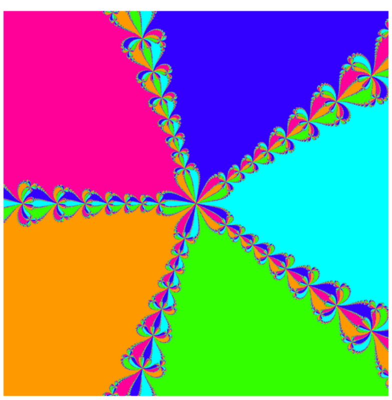
In 4 steps we expect to have $2^4 = 16$ digits correct. Having a fast method to compute roots is useful. For example, in computer graphics, where things can not be fast enough. We will explore a bit in the lecture how fast the method is.

14.3. If we have several roots, and we start at some point, to which root will the Newton method converge? Does it at all converge? This is an interesting question. It is also historically intriguing because it is one of the first cases, where "chaos" was observed at the end of the 19'th century.

Example: Find the Newton map in the case $f(x) = x^5 - 1$.
Solution

$$T(x) = x - \frac{x^5 - 1}{5x^4} \; .$$

14.4. If we look for roots in the complex like for $f(x) = x^5 - 1$ which has 5 roots in the complex plane, the "region of attraction" of each of the roots is a complicated set which we call the **Newton fractal**.




The Newton method is scrumtrulescent! (To quote from the "Inside the Actors Studio" at SNL)

Example: Lets compute $\sqrt{2}$ to 12 digits accuracy. We want to find a root $f(x) = x^2 - 2$. The Newton map is $T(x) = x - (x^2 - 2)/(2x)$. Lets start with $x = 1$.

$$\begin{aligned}
T(1) &= 1 - (1-2)/2 = 3/2 \\
T(3/2) &= 3/2 - ((3/2)^2 - 2)/3 = 17/12 \\
T(17/12) &= 577/408 \\
T(577/408) &= 665857/470832 \; .
\end{aligned}$$

This is already $1.6 \cdot 10^{-12}$ close to the real root! 12 digits, by hand!

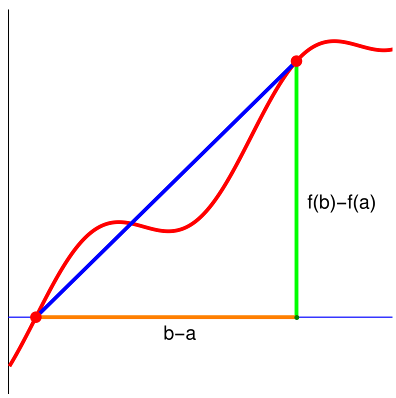
Example: To find the cube root of 10 we have to find a root of $f(x) = x^3 - 10$. The Newton map is $T(x) = x - (x^3 - 10)/(3x^2)$. If we start with $x = 2$, we get the following steps $2, 13/6, 3277/1521, 105569067476/49000820427$. After three steps we have a result which is already $2.2 \cdot 10^{-9}$ close to the root.

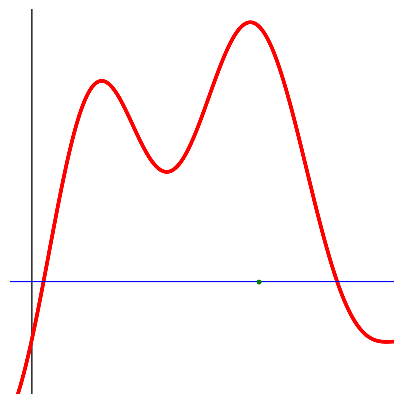
Example: Verify that the Newton map $T(x)$ in the case $f(x) = a(x-b)^n$ with $n > 0$ has the property that for the root $x = b$ is obtained.

14.5. Unlike the intermediate value theorem which applied for continuous functions, the mean value theorem involves derivatives. We assume therefore today that all functions are differentiable unless specified.

Mean value theorem: Any interval (a, b) contains a point x such that

$$f'(x) = \frac{f(b) - f(a)}{b - a} .$$





Here are a few examples which illustrate the theorem:

Example: Verify with the mean value theorem that the function $f(x) = x^2 + 4\sin(\pi x) + 5$ has a point where the derivative is 1.

Solution. Since $f(0) = 5$ and $f(1) = 6$ we see that $(f(1) - f(0))/(1 - 0) = 5$.

Example: A biker drives with velocity $f'(t)$ at position $f(b)$ at time b and at position a at time a. The value $f(b) - f(a)$ is the distance traveled. The fraction $[f(b) - f(a)]/(b - a)$ is the average speed. The theorem tells that there was a time when the bike had exactly the average speed.

14.6. Proof of the theorem: the function $h(x) = f(a) + cx$, where $c = (f(b) - f(a)/(b - a)$ also connects the beginning and end point. The function $g(x) = f(x) - h(x)$ has now the property that $g(a) = g(b)$. If we can show that for such a function, there exists x with $g'(x) = 0$, then we are done. By tilting the picture, we have reduced the statement to

Rolle's theorem: If $f(a) = f(b)$ then f has a critical point in (a, b).

Proof: If it were not true, then either $f'(x) > 0$ everywhere implying $f(b) > f(a)$ or $f'(x) < 0$ implying $f(b) < f(a)$.

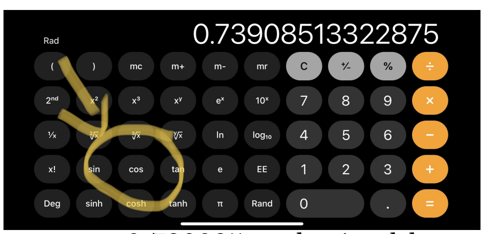
Example: Show that the function $f(x) = \sin(x) + x(\pi - x)$ has a critical point $[0, \pi]$.
Solution: The function is differentiable and nonnegative. It is zero at $0, \pi$. By Rolle's theorem, there is a critical point.

Example: Verify that the function $f(x) = 2x^3 + 3x^2 + 6x + 1$ has only one real root. **Solution:** There is at least one real root by the intermediate value theorem: $f(-1) = -4, f(1) = 12$. Assume there would be two roots. Then by Rolle's theorem there would be a value x where $g(x) = f'(x) = 6x^2 + 6x + 6 = 0$. But there is no root of g. [The graph of g minimum at $g'(x) = 6 + 12x = 0$ which is $1/2$ where $g(1/2) = 21/2 > 0$.]

Homework

Problem 14.1: Get the Newton map $T(x) = x - f(x)/f'(x)$ for:
a) $f(x) = (x-2)^2$
b) $f(x) = e^{5x}$
c) $f(x) = 2e^{-x^2}$
d) $f(x) = \cot(x)$.

Problem 14.2: The function $f(x) = \cos(x) - x$ has a root between 0 and 1. We get closer to the root by doing two Newton steps starting with $x = 1$.



Compare with the root $x = 0.739085...$ obtained by punching "cos" again and again

Problem 14.3: We want to find the square root of 102. We have to solve $\sqrt{102} = x$ or $f(x) = x^2 - 102 = 0$. Perform two Newton steps starting at $x = 10$.

Problem 14.4: Find the Newton step $T(x) = x - f(x)/f'(x)$ in the case $f(x) = 1/x$. What happens if you apply the Newton steps starting with $x = 1$? Does the method converge?

Problem 14.5: We look at the function $f(x) = x^{10} + x^4 - 20x$ on the positive real line. Use the **mean value theorem** on $(1, 2)$ to assure the there exists x, where $g(x) = f'(x) - [f(2) - f(1)] = f'(x) - 1018$. Now use one Newton step starting with 1.5 to find a solution to $g(x) = 0$