

# INTRODUCTION TO CALCULUS

MATH 1A

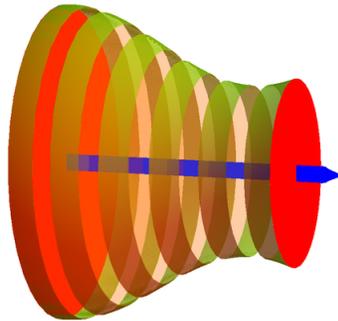
## Unit 21: Volume

### LECTURE

**21.1.** To compute the **volume of a solid**, we cut it into slices, where each slice is perpendicular to a given line  $x$ . If  $A(x)$  is the **area of the slice** and the body is enclosed between  $a$  and  $b$  then

$$V = \int_a^b A(x) dx$$

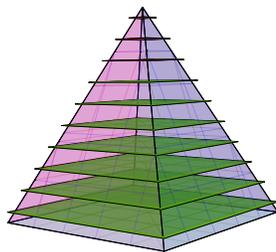
is the **volume**. Think of  $A(x)dx$  as the volume of a slice. The integral adds up.



**Example:** Compute the volume of a pyramid with square base length 2 and height 2. **Solution:** we can assume the pyramid is built over the square  $-1 \leq x \leq 1$  and  $-1 \leq y \leq 1$ . The cross section area at height  $h$  is  $A(h) = (2 - h)^2$ . Therefore,

$$V = \int_0^2 (2 - h)^2 dh = \frac{8}{3}.$$

This is base area 4 times height 2 divided by 3.



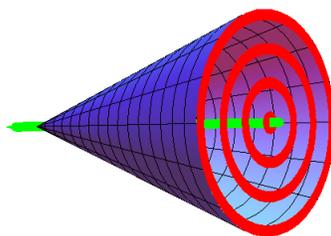
**Definition:** A **solid of revolution** is a surface enclosed by the surface obtained by rotating the graph of a function  $f(x)$  around the  $x$ -axis.

The area of the cross section at  $x$  of a solid of revolution is  $A(x) = \pi f(x)^2$ . The volume of the solid is  $\int_a^b \pi f(x)^2 dx$ .

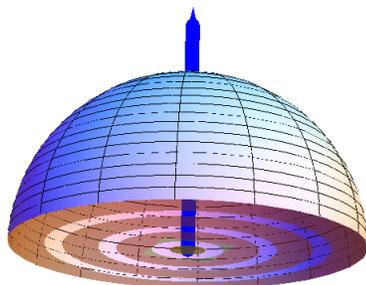
**Example:** Find the volume of a **round cone** of height 2 and where the circular base has the radius 1. **Solution.** This is a solid of revolution obtained by rotation the graph of  $f(x) = x/2$  around the  $x$  axes. The area of a cross section is  $\pi x^2/4$ . Integrating this up from 0 to 2 gives

$$\int_0^2 \pi x^2/4 dx = \frac{x^3}{4 \cdot 3} \Big|_0^2 = \frac{2\pi}{3}.$$

This is the height 2 times the base area  $\pi$  divided by 3.

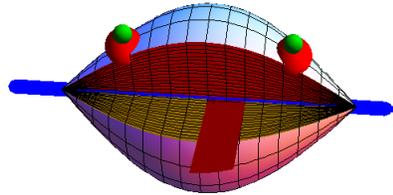


**Example:** Find the volume of a **half sphere** of radius 1. **Solution:** The area of the cross section at height  $h$  is  $\pi(1 - h^2)$ .



**Example:** If the function  $f(x) = \sin(x)$  is rotated around the  $x$  axes, we get a **lemon**. But now we cut out a slice of  $60 = \pi/3$  degrees as in the picture. Find the volume of the solid.

**Solution:** The area of a slice without the missing piece is  $\pi \sin^2(x)$ . The integral  $\int_0^\pi \sin^2(x) dx$  is  $\pi/2$  as derived in the lecture. Having cut out  $1/6$ 'th the area is  $(5/6)\pi \sin^2(x)$ . The volume is  $\int_0^\pi (5/6)\pi \sin^2(x) dx = (5/6)\pi^2/2$ .

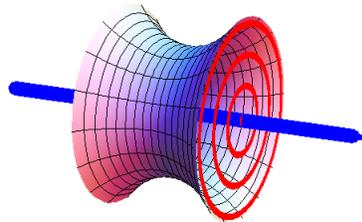


## Homework

**Problem 1:** Find the volume of the **paraboloid** for which the radius at position  $x$  is  $4 - x^2$  and  $x$  ranges from 0 to 2.

**Problem 2:** A **catenoid** is the surface obtained by rotating the graph of  $f(x) = \cosh(x) = (\exp(x) + \exp(-x))/2$  around the  $x$ -axes. We have seen that the graph of  $f$  is the chain curve, the shape of a hanging chain. Find the volume of of the solid enclosed by the catenoid between  $x = -2$  and  $x = 2$ .

**Hint.** You might want to check first the identity  $\cosh(x)^2 = (1 + \cosh(2x))/2$  using the definition  $\cosh(x) = (\exp(x) + \exp(-x))/2$ .



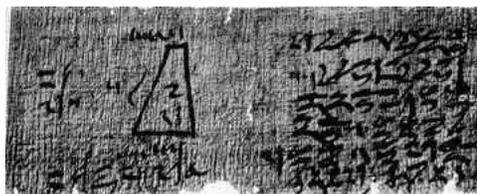
**Problem 3:** A **tomato** is given by  $z^2 + x^2 + 4y^2 = 1$ . If we slice perpendicular to the  $y$  axes, we get a circular slice  $z^2 + x^2 \leq 1 - 4y^2$  of radius  $\sqrt{1 - 4y^2}$ . Find the area of this slice, then determine the volume of the tomato.

**Problem 4: Archimedes** was so proud of his formula for the volume of a sphere that he wanted the formula displayed on his tomb stone. To derive the formula, he wrote the volume of a half sphere of radius 1 as the difference between the volume of a cylinder of radius 1 and height 1 and the volume of a cone of base radius 1 and height 1. Relate the cross section area of the cylinder-cone complement with the cross section area of the sphere to recover his argument! If stuck, draw in the sand, soak in the bath tub or eat a tomato salad. No credit is given for screaming “Eureka”.

**Problem 5:** Volumes were among the first quantities, Mathematicians wanted to measure and compute. One problem on **Moscow Egypt papyrus** dating back to 1850 BC explains the general formula  $h(a^2 + ab + b^2)/3$  for a **truncated pyramid** with base length  $a$ , roof length  $b$  and height  $h$ . Verify that if you slice such a **frustrum** at height  $x$ , the area is  $A(x) = (a + (b - a)x/h)^2$ . Now use this to compute the volume using calculus.

Here is the translated formulation from the papyrus: <sup>1</sup> <sup>2</sup>

**Remark:** ”You are given a truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. You are to square this 4 result 16. You are to double 4 result 8. You are to square 2, result 4. You are to add the 16, the 8 and the 4, result 28. You are to take one-third of 6 result 2. You are to take 28 twice, result 56. See it is 56. You will find it right”.



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<sup>1</sup>Howard Eves, Great moments in mathematics, Volume 1, MAA, Dolciani Mathematical Expositions, 1980, page 10

<sup>2</sup>Image Source: [http://www-history.mcs.st-and.ac.uk/HistTopics/Egyptian\\_papyri.html](http://www-history.mcs.st-and.ac.uk/HistTopics/Egyptian_papyri.html)