

# INTRODUCTION TO CALCULUS

MATH 1A

## Unit 28: Partial fractions

LECTURE

The method of partial fractions is not really about integration. It is about algebra. We have learned how to integrate polynomials like  $x^4 + 5x + 3$ . What about rational functions? We will see here that they are a piece of cake if you know a bit about algebra.



**28.1.** Lets see what we know already:

- We also know that integrating  $1/x$  gives  $\log(x)$ . We can for example integrate

$$\int \frac{1}{x-6} dx = \log(x-6) + C .$$

- We also have learned how to integrate  $1/(1+x^2)$ . It was an important integral:

$$\int \frac{1}{1+x^2} dx = \arctan(x) + C .$$

Using substitution, we can do more like

$$\int \frac{dx}{1+4x^2} = \int \frac{du/2}{1+u^2} = \arctan(u)/2 = \arctan(2x)/2 .$$

- We also know how to integrate functions of the type  $x/(x^2+c)$  using substitution. We can write  $u = x^2 + c$  and get  $du = 2xdx$  so that

$$\int \frac{x}{x^2+c} dx = \int \frac{1}{2u} du = \frac{\log(x^2+c)}{2} .$$

- Also functions  $1/(x+c)^2$  can be integrated using substitution. With  $x+c = u$  we get  $du = dx$  and

$$\int \frac{1}{(x+c)^2} dx = \int \frac{1}{u^2} du = -\frac{1}{u} + C = -\frac{1}{x+c} + C .$$

**28.2.** We would love to be able to integrate any rational function

$$f(x) = \frac{p(x)}{q(x)},$$

where  $p, q$  are polynomials. This is where **partial fractions come in**. The idea is to write a rational function as a sum of fractions we know how to integrate. The above examples have shown that we can integrate  $a/(x+c)$ ,  $(ax+b)/(x^2+c)$ ,  $a/(x+c)^2$  and cases, which after substitution are of this type.

**Definition:** The **partial fraction method** writes  $p(x)/q(x)$  as a sum of functions of the above type which we can integrate.

**28.3.** This is an algebra problem. Here is an important special case:

In order to integrate  $\int \frac{1}{(x-a)(x-b)} dx$ , write

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}.$$

and solve for  $A, B$ .

**28.4.** In order to solve for  $A, B$ , write the right hand side as one fraction again

$$\frac{1}{(x-a)(x-b)} = \frac{A(x-b) + B(x-a)}{(x-a)(x-b)}.$$

We only need to look at the nominator:

$$1 = Ax - Ab + Bx - Ba.$$

In order that this is true we must have  $A + B = 0$ ,  $Ab - Ba = 1$ . This allows us to solve for  $A, B$ .

## Examples

**Example:** To integrate  $\int \frac{2}{1-x^2} dx$  we can write

$$\frac{2}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$$

and integrate each term

$$\int \frac{2}{1-x^2} = \log(1+x) - \log(1-x).$$

**Example:** Integrate  $\frac{5-2x}{x^2-5x+6}$ . **Solution.** The denominator is factored as  $(x-2)(x-3)$ . Write

$$\frac{5-2x}{x^2-5x+6} = \frac{A}{x-3} + \frac{B}{x-2}.$$

Now multiply out and solve for  $A, B$ :

$$A(x-2) + B(x-3) = 5-2x.$$

This gives the equations  $A + B = -2$ ,  $-2A - 3B = 5$ . From the first equation we get  $A = -B - 2$  and from the second equation we get  $2B + 4 - 3B = 5$  so that  $B = -1$  and so  $A = -1$ . We have not obtained

$$\frac{5 - 2x}{x^2 - 5x + 6} = -\frac{1}{x - 3} - \frac{1}{x - 2}$$

and can integrate:

$$\int \frac{5 - 2x}{x^2 - 5x + 6} dx = -\log(x - 3) - \log(x - 2).$$

Actually, we could have got this one also with substitution. How?

**Example:** Integrate  $f(x) = \int \frac{1}{1-4x^2} dx$ . **Solution.** The denominator is factored as  $(1 - 2x)(1 + 2x)$ . Write

$$\frac{A}{1 - 2x} + \frac{B}{1 + 2x} = \frac{1}{1 - 4x^2}.$$

We get  $A = 1/4$  and  $B = -1/4$  and get the integral

$$\int f(x) dx = \frac{1}{4} \log(1 - 2x) - \frac{1}{4} \log(1 + 2x) + C.$$

**28.5.** There is a fast method to get the coefficients:

If  $a$  is different from  $b$ , then the coefficients  $A, B$  in

$$\frac{p(x)}{(x - a)(x - b)} = \frac{A}{x - a} + \frac{B}{x - b},$$

are

$$A = \lim_{x \rightarrow a} (x - a)f(x) = p(a)/(a - b), \quad B = \lim_{x \rightarrow b} (x - b)f(x) = p(b)/(b - a).$$

Proof. If we multiply the identity with  $x - a$  we get

$$\frac{p(x)}{(x - b)} = A + \frac{B(x - a)}{x - b}.$$

Now we can take the limit  $x \rightarrow a$  without peril and end up with  $A = p(a)/(a - b)$ .

**28.6.** Cool, isn't it? This **Hospital method** or **residue problem** saves time especially with many functions where we would a complicated system of linear equations would have to be solved.

Math is all about elegance. Avoid complicated methods if simple ones are available.

**28.7.** Here are examples:

**Example:** Find the anti-derivative of  $f(x) = \frac{2x+3}{(x-4)(x+8)}$ . **Solution.** We write

$$\frac{2x+3}{(x-4)(x+8)} = \frac{A}{x-4} + \frac{B}{x+8}$$

Now  $A = \frac{2 \cdot 4 + 3}{4 + 8} = 11/12$ , and  $B = \frac{2 \cdot (-8) + 3}{(-8 - 4)} = 13/12$ . We have

$$\frac{2x+3}{(x-4)(x+8)} = \frac{(11/12)}{x-4} + \frac{(13/12)}{x+8}.$$

The integral is

$$\frac{11}{12} \log(x-4) + \frac{13}{12} \log(x+8).$$

**Example:** Find the anti-derivative of  $f(x) = \frac{x^2+x+1}{(x-1)(x-2)(x-3)}$ . **Solution.** We write

$$\frac{x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Now  $A = \frac{1^2+1+1}{(1-2)(1-3)} = 3/2$  and  $B = \frac{2^2+2+1}{(2-1)(2-3)} = -7$  and  $C = \frac{3^2+3+1}{(3-1)(3-2)} = 13/2$ . The integral is

$$\frac{3}{2} \log(x-1) - 7 \log(x-2) + \frac{13}{2} \log(x-3).$$

## HOMEWORK

**Problem 28.1:** a)  $\int \frac{1}{x^2-14x+45} dx$   
b)  $\int \frac{2}{x^2-9} dx$

**Problem 28.2:**  $\int \frac{5dx}{4x^2+1}$ . (Does not need partial fractions!)

**Problem 28.3:**  $\int \frac{x^3-x+1}{x^2-1} dx$ .

**Problem 28.4:**  $\int \frac{23}{(x+2)(x-3)(x-2)(x+3)} dx$ . Use Hospital.

**Problem 28.5:**  $\int \frac{1}{(x+1)(x-1)(x+7)(x-3)} dx$ . Use Hospital.

**Hint for 3).** Subtract first a polynomial.