INTRODUCTION TO CALCULUS

MATH 1A

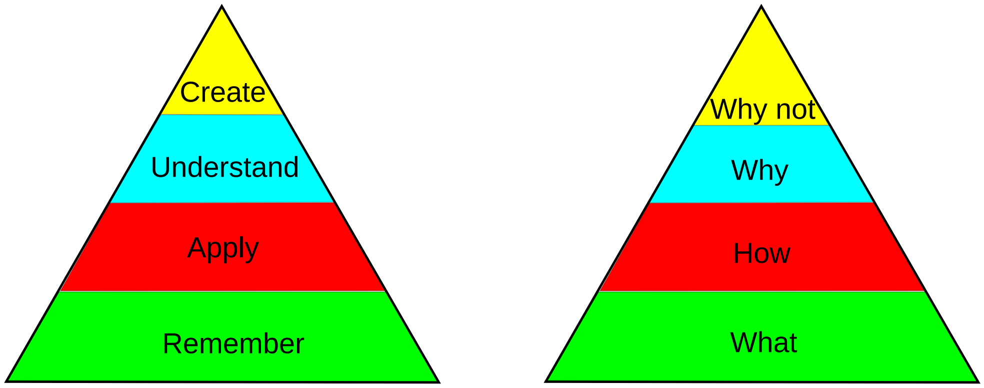
Unit 32: Calculus and AI

Lecture

About intelligence

32.1. Artificial intelligence has a lot of overlap with education. If one wants to build an artificial intelligence entity, one needs to teach it first or to teach it how to learn itself. In a project of 2013, we have learned a bit about this by programming an AI bot called Sofia. It gave us quite a bit of insight on **how humans learn**. Teaching definitions for example is very simple. A machine is very good in memorizing stuff. Also teaching algorithms is no problem for a machine if the task is communicated clearly. What turns out to be much more difficult is to "teach insight". How can one teach a machine for example to see what are "relevant" core principles, what is "important", what is "good taste". The last step is to teach being creative, or to discover new things. In that Sofia project, we boiled the process of learning and teaching down to the 4 questions "What, How, Why and Why not?". We realized later that this turned out to be a variant of the **Bloom taxonomy** which splits learning into **Factual**, **Conceptual**, **Procedural** and **Metacognitive** parts.

32.2. Here is an example in calculus. We teach the concept of derivative. What is a derivative? How does one compute a derivative? Why does one compute derivatives? Why does one not just compute something else? It is no problem to teach a machine that the derivative of $f(x)$ is the limit $(f(x+h)-f(x))/h$ in the limit $h \to 0$. It is also no problem to teach the machine to take the derivative of a function like $\sin(5x)x$. All computer algebra systems know that already. The question why one wants to compute derivatives is harder. But it is teachable too. We want to compute derivatives for example because we want to find maxima or minima of quantities. The question of how we can extend the concept of derivative or replace it with something else is harder. There are other notions in mathematics which have done so. In quantum calculus, one looks at $Df(x) = f(x+1)-f(x)$ for example, in other parts of math, one has notions of "derivations" which formalize operations satisfying the Leibniz rule $(fg)' = f'g+fg'$. In quantum mechanics, the derivative is essentially the "momentum operator" generating translation. To teach a machine to come up with such connections and concepts is much, much harder.



A modified Bloom taxonomy. A major change is to put the "apply" part before the "understanding" part. The rationale is that in almost all situations of learning, one first learns how to do something before knowing why it works.

Generating calculus problems

32.3. We ask "Sofia", our artificial intelligence teacher to automatically build worksheets or exam problems as well as solutions. In order to generate problems, we first must build **random functions**. When asked "give me an example of a function", the system should generate functions of some complexity:

> **Definition:** A **basic function** is a function from the 10 functions $\{\sin, \cos, \log, \exp, \tan, \text{sqt}, \text{pow}, \text{inv}, \text{sca}, \text{tra}\}$.

32.4. Here $\text{sqt}(x) = \sqrt{x}$ and $\text{inv}(x) = 1/x^k$ for a random integer k between -1 and -3, $\text{pow}(x) = x^k$ for a random integer k between 2 and 5. $\text{sca}(x) = kx$ is a scalar multiplication for a random nonzero integer k between -3 and 3 and $\text{tra}(x) = x + k$ translates for a random integer k between -4 and 4.

32.5. Second, we use addition, subtraction multiplication, division and composition to build more complicated functions:

> **Definition:** A **basic operation** is an operation from the list $\{f \circ g, f + g, f * g, f/g, f - g\}$.

32.6. The operation x^y is not included because it is equivalent to $\exp(x \log(y)) = \exp \circ (x \cdot \log)$. We can now build functions of various complexities:

> **Definition:** A **random function** of complexity n is obtained by taking n random basic functions $f_1, \dots, f_n$, and n random basic operators $\oplus_1, \dots, \oplus_n$ and forming $f_n \oplus_n f_{n-1} \oplus_{n-1} \cdots \oplus_2 f_1 \oplus_1 f_0$ where $f_0(x) = x$ and where we start forming the function from the right.

Example: **Visitor:** "Give me an easy function": Sofia looks for a function of complexity one: like $x\tan(x)$, or $x+\log(x)$, or $-3x^2$, or $x/(x-3)$.

Example: **Visitor:** "Give me a function": Sofia returns a random function of complexity two: $x\sin(x)-\tan(x)$, or $-e^{\sqrt{x}}+\sqrt{x}$ or $x\sin(x)/\log(x)$ or $\tan(x)/x^4$.

Example: **Visitor:** "Give me a difficult function": Sofia builds a random function of complexity four like $x^4e^{-\cos(x)}\cos(x)+\tan(x)$, or $x-\sqrt{x}-e^x+\log(x)+\cos(x)$, or $(1+x)(x\cot(x)-\log(x))/x^2$, or $(-x+\sin(x+3)-3)\csc(x)$

32.7. Now, we can build a random calculus problem. To give you an idea, here are some templates for integration problems:

> **Definition:** A **random integration problem** of complexity n is a sentence from the sentence list { "Integrate $f(x)=F(x)$", "Find the anti derivative of $F(x)$", "What is the integral of $f(x)=F(x)$?", "You know the derivative of a function is $f'(x)=F(x)$. Find $f(x)$." }, where F is a random function of complexity n.

Example: **Visitor** "Give me a differentiation problem". **Sofia:** Differentiate $f(x)=x\sin(x)-\frac{1}{x^2}$. The answer is $\frac{2}{x^3}+\sin(x)+x\cos(x)$.

Example: **Visitor**: "Give me a difficult integration problem". **Sofia**: Find f if $f'(x)=\frac{1}{x}+\left(3\sin^2(x)+\sin(\sin(x))\right)\cos(x)$. The answer is $\log(x)+\sin^3(x)-\cos(\sin(x))$.

Example: **Visitor**: "Give me an easy extremization problem". **Sofia**: Find the extrema of $f(x)=x/\log(x)$. The answer is $x=e$.

Example: **Visitor**: Give me an extremization problem". **Sofia**: Find the maxima and minima of $f(x)=x-x^4+\log(x)$. The extrema are

$$-\frac{\sqrt{\left(9+\sqrt{3153}\right)^{2/3}-8\sqrt[3]{6}}+\sqrt{8\sqrt[3]{6}-\left(9+\sqrt{3153}\right)^{2/3}\left(1+6\sqrt{\frac{2}{9+\sqrt{3153}-8\sqrt[3]{6\left(9+\sqrt{3153}\right)}}}\right)}}{2 2^{5/6}\sqrt[3]{3}\sqrt[6]{9+\sqrt{3153}}}.$$

The last example shows the perils of random generation. Even so the function had decent complexity, the solution was difficult. Solutions can even be transcendental. This is not a big deal: just generate a new problem. By the way, all the above problems and solutions have been generated by Sofia. The dirty secret of calculus books is that there are maybe a thousand different type of questions which are usually asked. This is a reason why textbooks have become boring clones of each other and companies like "Aleks", "Demidec" etc exist which constantly mine the web and course sites like this and homework databases like "webwork" which contain thousands of pre-compiled problems in which randomness is already built in.

> Automated problem generation is the "fast food" of teaching and usually not healthy. But like "fast food" has evolved, we can expect more and more computer assisting in calculus teaching.

Be assured that for this course, problems have been written by hand. (Sometimes Mathematica is used to see whether answers are reasonable). Handmade problems can sometimes a bit "rough" but can be more interesting. Still, you can see a worksheet which has been generated entirely by a program.

Homework

Problem 32.1: Lets build a differentiation problem by combining log and sin and exp. Differentiate all of the 6 combinations $\log(\sin(\exp(x)))$, $\log(\exp(\sin(x)))$,$\exp(\log(\sin(x)))$, $\exp(\sin(\log(x)))$, $\sin(\log(\exp(x)))$ and $\sin(\exp(\log(x)))$.

Problem 32.2: Four of the 6 combinations of log and sin and exp can be integrated as elementary functions.
a) Find these cases
b) Do these integrals.

Problem 32.3: From the 10 functions f and 10 functions g and 5 operations, we can build 500 functions. Some can not be integrated. An example is $\exp(\sin(x))$. Find 4 more which can not be integrated by you and computer algebra.

Problem 32.4: One of the most difficult things to teach is creativity. Lets try to be creative. Build an extremization problem which is applied. Here is an example (of course your example should be different than this): **A common theme for extrema are area and length. Invent an extremum problem involving an isoscele triangle. It should be of the form: "Maximize the area ... ". Now solve the problem.**

Problem 32.5: Be creative: a) Create an area problem for a region in the plane which has not appeared in lecture, homework or exam.
b) Create a volume problem for a surface of revolution. The problem should not have appeared yet in lecture, homework or exams. Make the problem so that it can be solved. Now solve your problem.

Oliver Knill, knill@math.harvard.edu, Math 1a, Harvard College, Spring 2020