INTRODUCTION TO CALCULUS

MATH 1A

Lecture 33: Calculus and Music

Music is a function

33.1. Calculus is relevant in music because a music piece is just a **function**. If you feed a loudspeaker the function $f(t)$ the membrane gets displaced by $f(t)$. The pressure variations in the air are sound waves then reach your ear, where your ear drum oscillates as such. Plotting and playing works the same way. In Mathematica, we can play a function by replacing "Plot" with "Play" and say with

```
Play[ Sin[2Pi 1000 x^2],{x,0,10}]
```

33.2. While function f contains all the information about the music piece, the computer needs to store this as data. One possibility is in a ".WAV" file, which contains sampled values of the function usually with a sample rate of 44100 readings per second. Since our ear does not hear frequencies larger than 20'000 KHz, a sampling rate of 44.1K is good enough by a **theorem of Nyquist-Shannon**. More sophisticated storage possibilities exist. A .MP3 file for example encodes the function in a compressed way. To get from the sample values $f(k)$ the function back, the **Whittaker-Shannon interpolation formula**

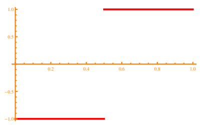
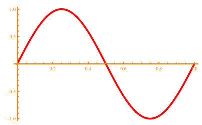
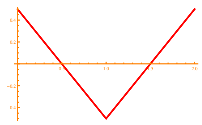
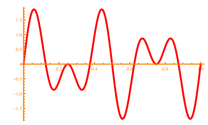
$$f(t) = \sum_{k=1}^{n} f(k)\mathrm{sinc}(t+k)$$

can be used. It involves the sinc function $\mathrm{sinc}(x) = \sin(x)/x$ we have seen early in the course.

The wave form and hull

33.3. A periodic signal is the building block of sound. Assume $g(x)$ is a 2π-periodic function, we can generate a sound of 440 Hertz when playing the function $f(x) = g(440 \cdot 2\pi x)$. If the function does not have a smaller period, then we hear the A **tone**. It is a tone with 440 Hertz.

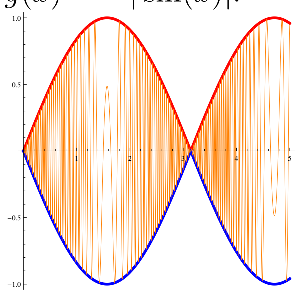
Definition: A periodic function g is called a **wave form**.

33.4. The wave form makes up the **timbre** of a sound which allows to model music instruments with macroscopic terms like **attack**, **vibrato**, **coloration**, **noise**, **echo**, **reverberation** and other characteristics.

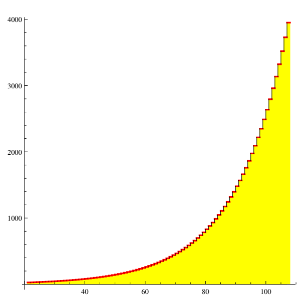
Definition: The **upper hull function** is defined as the interpolation of successive local maxima of f. The **lower hull function** is the interpolation of the local minima.

33.5. For the function $f(x) = \sin(100x)$ for example, the upper hull function is $g(x) = 1$ and the lower hull function is $g(x) = -1$. For $f(x) = \sin(x)\sin(100x)$ the upper hull function is approximately $g(x) = |\sin(x)|$ and the lower hull function is approximately $g(x) = -|\sin(x)|$.



We can not hear the actual function $f(x)$ because the function changes too fast that we can notice individual vibrations. But we can hear the hull function. We can hear **large scale amplitude** changes like **creshendi** or **diminuendi** or a **vibrato**. When playing two frequencies which are close, you hear **interference**.

The scale



33.6. Western music uses a discrete set of frequencies. This scale is based on the **exponential function**. The frequency f is an exponential function of the scale s. On the other hand, if the frequency is known then the scale number is a logarithm. This is a nice application of the logarithm:

Definition: A frequency f has the **Midi number** $s = 69 + 12 \cdot \log_2(f/440)$. The **piano scale function** or **midi function** gives back $f(s) = 440 \cdot 2^{(s-69)/12}$.

33.7. The Midi tone $s = 100$ for example gives $f = 2637.02$ Hertz.

The **piano scale function** $f(s) = 440 \cdot 2^{(s-69)/12}$ is an exponential function $f(s) = be^{as}$ which satisfies $f(s+12) = 2f(s)$.

```
midifrequency[m_] := N[440 2^((m - 69)/12)]
```

33.8. The classical piano covers the 88 Midi tone scale from 21 to 108. It ranges from $f = 27.5Hz$, the sub-contra-octave A, to the highest $f = 4186.01Hz$, the 5-line octave C.

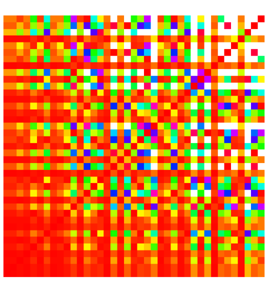
33.9. Filters: a function can be written as a sum of sin and cos functions. Our ear does this so called **Fourier decomposition** automatically. We can so hear melodies, filter out part of the music and hum it.
Pitch and autotune: it is possible to filter out frequencies and adapt their frequency. The popular filter **autotune** moves the frequencies around correcting wrong singing. If 440 Hertz (A) and 523.2 Hertz (C) for example were the only allowed frequencies, the filter would change a function $f(x) = \sin(2\pi 441x) + 4\cos(2\pi 521x)$ to $g(x) = \sin(2\pi 440x) + 4\cos(2\pi 523.2x)$. **Rip and remix**: if f and g are two songs, we can build the average $(f+g)/2$. A composer does this using **tracks**. Different instruments are recorded independently and then mixed together. A guitar $g(t)$, a voice $v(t)$ and a piano $p(t)$ together can form $f(t) = ag(t)+bv(t)+c(p(t)$ with suitably chosen constants a, b, c. **Reverberate and echo**: if f is a song and h is some time interval, we can look at $g(x) = Df(x) = [f(x+h)-f(x)]/h$. For small h, like $h = 1/1000$ the song does not change much because hearing $\sin(kx)$ or $\cos(kx)$ produces the same song. However, for larger h, one can get **reverberate** or **echo** effects.

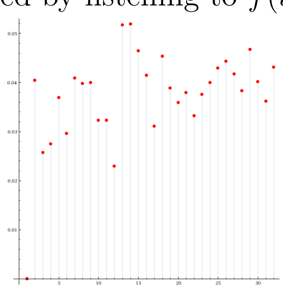
Other math relations

33.10. Mathematics and music have a lot of overlap. Besides wave form analysis and music manipulation operations and symmetry, there are **encoding and compression problems**. A **Diophantine problem** is the question how well a frequency can be approximated by rationals. Why is the **chromatic scale** based on $2^{1/12}$ so effective? **Indian music** for example uses **micro-tones** and a scale of 22. The 12-tone scale has the property that many powers $2^{k/12}$ are close to rational numbers. This can be quantified with the **scale fitness**

$$M(n) = \sum_{k=1}^{n} \min_{\mathrm{p,q}} |2^{k/n} - \frac{p}{q}| G(p,q)$$

where $G(n,m)$ is Euler's **gradus suavis** (="degree of pleasure") defined as $G(n,m) = 1 + E(nm/\gcd(n,m))$ with **Euler gradus** $E(n) = \sum_{p|n} e(p)(p-1)$. The sum runs over all prime factors p of n and $e(p)$ is the multiplicity. The figure below shows that $n = 12$ has the best $M(n)$. The 2 could be replaced too. The **Stockhausen scale** uses $5^{k/25}$. You can hear it $f(t) = \sin(2\pi t 100 \cdot 5^{[t]/25})$, where $[t]$ is the largest integer smaller than t. The familiar **12-tone scale** can be admired by listening to $f(t) = \sin(2\pi t 100 \cdot 2^{[t]/12})$.





Example: The perfect fifth $3/2$ has the gradus suavis $1 + E(6) = 1 + 2 = 3$ which is the same than the perfect fourth $4/3$ for which $1 + E(12) = 1 + (2 - 1)(3 - 1)$. You can listen to the perfect fifth $f(x) = \sin(1000x) + \sin(1500x)$ or the perfect fourth $\sin(1000x) + \sin(1333x)$ and here is a function representing an **accord** with four notes $\sin(1000x) + \sin(1333x) + \sin(1500x) + \sin(2000x)$.

Homework

Problem 33.1: Modulation. Draw the hull function of the following functions.
a) $f(x) = \cos(200x) - \cos(201x)$
b) $f(x) = \cos(x) + \cos(\tan(1000\sqrt{x}))$
c) $f(x) = \sqrt{x}\cos(10000x)$
d) $f(x) = \cos(x)\sin(e^{2x})/2$

Here is how to play a function with Mathematica or Wolfram alpha:

```
Play[Cos[x] Sin[Exp[2 x]]/x, {x, 0,9}]
play  sin(1000 x)
```

Problem 33.2: Amplitude modulation (AM): If you listen to $f(x) = \cos(x)\sin(1000x)$ you hear an amplitude change. Draw the hull function. How many increase in amplitudes to you hear in 10 seconds?

Problem 33.3: Other tonal scales, Midi number: As a creative musician, you create your own tonal scale. You decide to take the 8'th root of 3 as your basic frequency change from one tone to the next.
a) After how many tonal steps has the frequency f tripled?
b) Build the midi function and its inverse for your tonal scale.

Problem 33.4: a) What is the frequency of the Midi number $s = 22$?
b) Which midi number belongs to the frequency $f = 2060 Herz$?

Problem 33.5: Gradus Suavitatis. a) What is the gradus suavitatis of $25/64$? b) What is the gradus suavitatis of $5904/65536$?

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