MATH 1A – PROBLEM SET 10.

EVA GOLUZA

* PROBLEM 10.1

A) $f(x) = \sin(\log(x))$

$$f'(x) = \frac{d}{dx}(\sin(\log(x)))$$

$$f'(x) = \frac{d}{du}\sin(u) \cdot \frac{d}{dx}(u) \quad \text{when } u = \log(x) \qquad \frac{d}{du}(\sin(u)) = \cos(u)$$

$$f'(x) = \cos(\log(x)) \cdot \frac{d}{dx}(\log(x))$$

$$f'(x) = \cos(\log(x)) \cdot \frac{d}{du}(\log(u)) \cdot \frac{d}{dx}(u) \quad \text{when } u = x \qquad \frac{d}{du}(\log(u)) = \frac{1}{u}$$

$$f'(x) = \cos(\log(x)) \cdot \frac{1}{u}$$

$$\boxed{f'(x) = \frac{\cos(\log(x))}{x}}$$

B) $f(x) = \tan(x^{11})$

$$f'(x) = \frac{d}{dx}(\tan(x^{11}))$$

$$f'(x) = \frac{d}{du}(\tan(u)) \cdot \frac{d}{dx}(u) \quad \text{when } u = x^{11} \qquad \frac{d}{du}(\tan(u)) = \sec^2(u)$$

$$f'(x) = \sec^2(u) \cdot \frac{d}{dx}(u)$$

$$f'(x) = \sec^2(x^{11}) \cdot \frac{d}{dx}(x^{11})$$

$$f'(x) = \sec^2(x^{11}) \cdot 11x^{10}$$

$$\boxed{f'(x) = 11x^{10}\sec^2(x^{11})}$$

C) $f(x) = \exp\left(\frac{1}{1+x^2}\right)$

$$f'(x) = \frac{d}{dx}\left(\exp\left(\frac{1}{1+x^2}\right)\right)$$

$$f'(x) = \frac{d}{dx}\left(\sqrt[1+x^2]{e}\right)$$

$$f'(x) = \frac{d}{du}(e^u)\ \frac{d}{dx}(u) \quad \text{where } u = \frac{1}{x^2+1} \qquad \frac{d}{du}(e^u) = e^u$$

$$f'(x) = e^{\frac{1}{x^2+1}} \cdot \frac{d}{dx}\left(\frac{1}{1+x^2}\right)$$

$$f'(x) = e^{\frac{1}{x^2+1}} \cdot \frac{0\cdot(1+x^2) - 2x\cdot 1}{(1+x^2)^2}$$

$$f'(x) = e^{\frac{1}{x^2+1}} \cdot \left(\frac{-2x}{(1+x^2)^2}\right)$$

$$\boxed{f'(x) = -2x\frac{\sqrt[1+x^2]{e}}{(1+x^2)^2}}$$

D) $f(x) = (3+\sin(x))^{-5}$

$$f(x) = \frac{1}{(3+\sin(x))^5}$$

$$f'(x) = \frac{d}{dx}\left(\frac{1}{(3+\sin(x))^5}\right)$$

$$f'(x) = \frac{d}{du}\left(\frac{1}{u^5}\right)\cdot\frac{d}{dx}(u) \quad \text{where } u = \sin(x)+3 \qquad \frac{d}{du}\left(\frac{1}{u^5}\right) = -\frac{5}{u^6}$$

$$f'(x) = -\frac{5}{u^6}\cdot\frac{d}{dx}(u)$$

$$f'(x) = -\frac{5}{(3+\sin(x))^6}\cdot\frac{d}{dx}(3+\sin(x))$$

$$f'(x) = -\frac{5}{(3+\sin(x))^6}\cdot\cos(x)$$

$$\boxed{f'(x) = -\frac{5\cos(x)}{(3+\sin(x))^6}}$$

* PROBLEM 10.2

A) $f(x) = -x\log(x)$

$$f'(x) = \frac{d}{dx}(-x\log(x))$$

$$f'(x) = -\left(\frac{d}{dx}(x\log(x))\right)$$

$$f'(x) = -\left(1\cdot\log(x) + x\cdot\frac{1}{x}\right)$$

$$f'(x) = -(\log(x)+1)$$

$$f'(x) = -\log(x) - 1$$
$$f'(1) = 0 - 1$$
$$f'(1) = -1$$

$$\boxed{f'(1) = -1}$$

B) $f(x) = \sqrt{x^5+1}$

$$f'(x) = \frac{d}{dx}\left(\sqrt{x^5+1}\right)$$

$$f'(x) = \frac{d\sqrt{u}}{du}\cdot\frac{d}{dx}(u) \quad \text{where } u = x^5+1$$

$$f'(x) = \frac{1}{2\sqrt{u}}\cdot\frac{d}{dx}(u)$$

$$f'(x) = \frac{1}{2\sqrt{x^5+1}}\cdot\frac{d}{dx}(1+x^5)$$

$$f'(x) = \frac{1}{2\sqrt{x^5+1}}\cdot 5x^4$$

$$f'(x) = \frac{5x^4}{2\sqrt{x^5+1}}$$

$$f'(1) = \frac{5\cdot 1}{2\sqrt{1+1}} = \frac{5}{2\sqrt{2}}$$

$$\boxed{f'(1) = \frac{5\sqrt{2}}{4}}$$

C) $f(x) = (1+x^2+x^4)^{100}$

$$f'(x) = \frac{d}{dx}\left((1+x^2+x^4)^{100}\right)$$

$$f'(x) = \frac{d}{du}(u^{100}) \cdot \frac{d}{dx}(u) \quad \text{where } u = 1+x^2+x^4$$

$$f'(x) = 100u^{99} \cdot \frac{d}{dx}(u)$$

$$f'(x) = 100(1+x^2+x^4)^{99} \cdot (2x+4x^3)$$
$$f'(1) = 100(1+1+1)^{99} \cdot (2+4)$$
$$f'(1) = 100(3)^{99} \cdot 6$$
$$f'(1) = 600(3)^{99}$$
$$f'(1) = 200 \cdot 3 \cdot 3^{99}$$
$$f'(1) = 200 \cdot 3^{100}$$

$$\boxed{f'(1) = 200 \cdot 3^{100} = 1.03 \cdot 10^{50}}$$

D) $f(x) = \dfrac{5x^4}{2\sqrt{x^5+1}}$

$$f'(x) = \frac{d}{dx}\left(\frac{5x^4}{2\sqrt{x^5+1}}\right)$$

$$f'(x) = \frac{5}{2} \cdot \frac{d}{dx}\left(\frac{x^4}{\sqrt{x^5+1}}\right)$$

$$f'(x) = \frac{5}{2} \cdot \frac{\frac{d}{dx}(x^4)\cdot\left(\sqrt{x^5+1}\right) - x^4 \cdot \frac{d}{dx}\left(\sqrt{x^5+1}\right)}{\left(\sqrt{x^5+1}\right)^2}$$

$$\frac{d}{dx}(x^4) = 4x^3 \qquad \frac{d}{dx}\left(\sqrt{x^5+1}\right) = \frac{d}{du}(\sqrt{u})\,\frac{d}{dx}(u) \quad \text{where } u = x^5+1$$

$$= \frac{1}{2\sqrt{u}} \cdot \frac{d}{dx}(u) = \frac{1}{2\sqrt{x^5+1}} \cdot (5x^4)$$

$$f'(x) = \frac{5}{2} \cdot \frac{4x^3 \cdot \sqrt{x^5+1} - x^4 \cdot \frac{5x^4}{2\sqrt{x^5+1}}}{\left(\sqrt{x^5+1}\right)^2}$$

$$\boxed{f'(1) = \frac{55\sqrt{2}}{16} \simeq 4.86}$$

$$f'(1) = \frac{5}{2} \cdot \frac{4\sqrt{2} - \frac{5}{2\sqrt{2}}}{2} = \frac{5}{2} \cdot \frac{4\sqrt{2} - \frac{5\sqrt{2}}{4}}{2} = \frac{5}{2} \cdot \frac{\frac{11\sqrt{2}}{4}}{\frac{2}{1}} = \frac{5}{2} \cdot \frac{11\sqrt{2}}{8} = \frac{55\sqrt{2}}{16}$$

* PROBLEM 10.3

A) $f(x) = \frac{1}{x}$

$xf(x) = 1$

$$\frac{d}{dx}(xf(x)) = 0$$

$$x \cdot f'(x) + 1 \cdot f(x) = 0$$

$$x \cdot f'(x) = -\frac{1}{x}$$

$$\boxed{f'(x) = -\frac{1}{x^2}}$$

B) $f(x) = \operatorname{arcot}(x)$

$\cot(\operatorname{arcot}(x)) = x$

$$\frac{d}{dx}\left[\cot(\operatorname{arcot}(x))\right] = 1$$

$$-\csc^2 x(\operatorname{arcot}(x)) \cdot \frac{d}{dx}(\operatorname{arcot}(x)) = 1$$

$\sin^2 x + \cos^2 x = 1$

$1 + \cot^2 x = \frac{1}{\sin^2 x}$

$1 + \cot^2 x = \csc^2 x$

$\cot^2(x) \cdot \operatorname{arcot}(x) = x^2$

$$\frac{d}{dx}(\operatorname{arcot}(x)) = -\frac{1}{\csc^2 x(\operatorname{arcot}(x))}$$

$$\frac{d}{dx}(\operatorname{arcot}(x)) = -\frac{1}{1 + \cot^2(x)(\operatorname{arcot}(x))}$$

$$\boxed{\frac{d}{dx}(\operatorname{arcot}(x)) = -\frac{1}{1 + x^2}}$$

* PROBLEM 10.4

A) $f(x)^2 = x$ $\sqrt{x}$

$$\frac{d}{dx} f(x)^2 = \frac{d}{dx}(x)$$

$$2 \cdot f(x) \cdot \frac{d}{dx} f(x) = 1$$

$$2f(x) \cdot f'(x) = 1$$

$$f'(x) = \frac{1}{2f(x)}$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{x}}}$$

B) $x^{m/n}$

$$f(x)^n = x^m$$

$$\frac{d}{dx}(f(x)^n) = \frac{d}{dx}(f(x)^m)$$

$$n \cdot f(x)^{n-1} \cdot \frac{d}{dx}(f(x)) = mx^{m-1}$$

$$n \cdot f(x)^{n-1} \cdot f'(x) = mx^{m-1}$$

$$f'(x) = \frac{m}{n} \cdot \frac{x^{m-1}}{f(x)^{n-1}}$$

$$f'(x) = \frac{m}{n} \cdot \frac{x^{m-1}}{((x)^{m/n})^{n-1}}$$

$$f'(x) = \frac{m}{n} \cdot \frac{x^{m-1}}{x^{\frac{m(n-1)}{n}}}$$

$$f'(x) = \frac{m}{2} \cdot x^{m-1-\left[\frac{m(n-1)}{n}\right]}$$

$$f'(x) = \frac{m}{2} \cdot x^{m-1-\left[\frac{mn-m}{n}\right]}$$

$$f'(x) = \frac{m}{2} \cdot x^{m-1-\left[\frac{m\cancel{n}}{\cancel{n}} - \frac{m}{n}\right]}$$

$$f'(x) = \frac{m}{2} \cdot x^{\cancel{m}-1-\cancel{m}+\frac{m}{n}}$$

$$\boxed{f'(x) = \frac{m}{2} \cdot x^{\left(\frac{m}{n}-1\right)}}$$

*PROBLEM 10.5

A) $f(x) = \cosh(x)$

$$f(x) = \frac{\exp(x) + \exp(-x)}{2}$$

$f^{-1}(x) = \operatorname{arccosh}(x)$

$\cosh(\operatorname{arccosh}(x)) = x$

$$\frac{d}{dx}(\cosh(\operatorname{arcosh}(x)) = \frac{d}{dx}(x)$$

$$\frac{d}{dx}(\cosh(\operatorname{arccosh}(x)) = 1$$

$$\sinh(\operatorname{arcosh}(x)) \cdot \frac{d}{dx}(\operatorname{arcosh}(x)) = 1$$

$$\frac{d}{dx}(\operatorname{arcosh}(x)) = \frac{1}{\sinh(\operatorname{arcosh}(x))}$$

$$\frac{d}{dx}(\operatorname{arccosh}(x)) = \frac{1}{\sqrt{\cosh^2(\operatorname{arccosh}(x)) - 1}}$$

$$\frac{d}{dx}(\operatorname{arccosh}(x)) = \boxed{\frac{1}{\sqrt{x^2 - 1}}}$$

$$\frac{d}{dx}(\cosh(x)) = \sinh(x)$$

$\cosh^2(x) - \sinh^2(x) = 1$

$-\sinh^2(x) = 1 - \cosh^2(x)$

$\sinh^2(x) = \cosh^2(x) - 1$

$\sinh(x) = \sqrt{\cosh^2(x) - 1}$

$\cosh(\operatorname{arcosh}(x)) = x$

$\cosh^2(\operatorname{arccosh}(x)) = x^2$

B)

$$f(x) = \sinh(x)$$

$$f(x) = \frac{\exp(x) - \exp(-x)}{2}$$

$$f^{-1}(x) = \operatorname{arcsinh}(x)$$

$$\sinh(\operatorname{arcsinh}(x)) = x$$

$$\frac{d}{dx}(\sinh(\operatorname{arcsinh}(x)) = \frac{d}{dx}(x)$$

$$\frac{d}{dx}(\sinh(\operatorname{arcsinh}(x)) = 1$$

$$\cosh(\operatorname{arcsinh}(x)) \cdot \frac{d}{dx}(\operatorname{arcsinh}(x)) = 1$$

$$\frac{d}{dx}(\operatorname{arcsinh}(x)) = \frac{1}{\cosh(\operatorname{arcsinh}(x))}$$

$$\frac{d}{dx}(\operatorname{arcsinh}(x)) = \frac{1}{\sqrt{\sinh^2(\operatorname{arcsinh}(x)) + 1}}$$

$$\frac{d}{dx}(\operatorname{arcsinh}(x)) = \boxed{\frac{1}{\sqrt{x^2 + 1}}}$$

$$\frac{d}{dx}(\sinh(x) = \cosh(x)$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\cosh^2(x) = 1 + \sinh^2(x)$$

$$\cosh(x) = \sqrt{1 + \sinh^2(x)}$$

$$\sinh(\operatorname{arcsinh}(x)) = x$$

$$\sinh^2(\operatorname{arcsinh}(x)) = x^2$$