MATH 1A – PROBLEM SET (11) EVA GOLUZA

* PROBLEM 11.1

A) $f(x) = x^6 - 3x^2$

$f'(x) = 6x^5 - 6x$

$f'(x_0) = 0$

$6x^5 - 6x = 0$

$$\boxed{\begin{array}{l} x = -1 \\ x = 0 \\ x = 1 \end{array}}$$

B) $f(x) = 4\sin(\pi x) + 3$

$f'(x) = 4\pi\cos(\pi x)$

$f'(x_0) = 0$

$4\pi\cos(\pi x) = 0$

$\cos(\pi x) = 0$

$x = \pi - \frac{1}{2}$ $\boxed{x_0 = k\pi - \frac{1}{2}}$

C) $f(x) = \exp(-x^2)\, x^3$

$f(x) = e^{-x^2} \cdot x^3$

$f'(x) = \frac{d}{dx}(e^{-x^2}) \cdot x^3 + e^{-x^2} \cdot \frac{d}{dx}(x^3)$

$f'(x) = e^{-x^2} \cdot (3x^2 - 2x^4)$

$f'(x_0) = 0$

$e^{-x^2} \cdot (3x^2 - 2x^4) = 0$

$x = 0$

$x = -\sqrt{\frac{3}{2}}$

$x = \sqrt{\frac{3}{2}}$

$$\boxed{\begin{array}{l} x = 0 \\ x = -\sqrt{\frac{3}{2}} \\ x = \sqrt{\frac{3}{2}} \end{array}}$$

D) $f(x) = \sin(\cos(\pi x))$

$f'(x) = \frac{d}{dx}(\sin(\cos(\pi x))$

$f'(x) = \frac{d}{du}(\sin(u)) \cdot \frac{d}{dx}(u)$ $u = \cos(\pi x)$

$f'(x) = \cos(\cos(\pi x))\left(\frac{d}{dx}(\cos \pi x)\right)$

$f'(x) = \cos(\cos(\pi x)) \cdot \frac{d}{du}(\cos(u))$ $u = \pi x$

$f'(x) = -\pi\sin(\pi x)\cos(\cos(\pi x))$

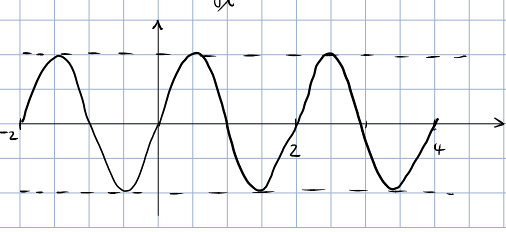
$f'(x_0) = 0$

$-\pi\sin(\pi x)\cos(\cos(\pi x)) = 0$

$-\pi\sin(\pi x) = 0$

$\sin(\pi x) = 0$

$\text{period} = \frac{2\cancel{\pi}}{\cancel{\pi}} = 2$



$\boxed{x = 0 \pm 1 \cdot n}$

* PROBLEM 11.2

A) $f(x) = x\log(x)$ $\quad x > 0$

$$f'(x) = \frac{d}{dx}(x) \cdot \log(x) + x \cdot \frac{d}{dx}(\log(x))$$

$f'(x) = \log(x) + 1$ $\qquad f''(x) = \frac{1}{x}$

$f'(x_0) = 0$

$\log(x) + 1 = 0$

$x = \frac{1}{e}$ $\qquad f''\left(\frac{1}{e}\right) = \frac{\frac{1}{1}}{\frac{1}{e}} = e$

$\boxed{e > 0 \Rightarrow x = \frac{1}{e} \text{ IS A LOCAL MINIMUM}}$

B) $f(x) = \frac{1}{(1+x^2)}$

$$f'(x) = \frac{\frac{d}{dx}(1) \cdot (1+x^2) - 1 \cdot \frac{d}{dx}(1+x^2)}{(1+x^2)^2}$$

$f'(x) = \frac{-1 \cdot 2x}{(1+x^2)^2} = -\frac{2x}{(1+x^2)^2}$ $\qquad f''(x) = \frac{d}{dx}\left(\frac{-2x}{(1+x^2)^2}\right)$

$f'(x_0) = 0$

$\frac{-2x}{(1+x^2)^2} = 0$

$-2x = 0$

$x = 0$

$$f''(x) = \frac{\frac{d}{dx}(-2x) \cdot (1+x^2)^2 + 2x \cdot \frac{d}{dx}((1+x^2)^2)}{((1+x^2)^2)^2}$$

$$f''(x) = \frac{-2(1+x^2)^2 + 2x \cdot (2+2x^2)}{(1+x^2)^4}$$

$$f''(x) = \frac{-2(1+x^2) + 4x(1+x^2)}{(1+x^2)^4}$$

$$f''(x) = \frac{(6x^2-2)\cancel{(1+x^2)}}{(1+x^2)^{\cancel{4}}} = \frac{6x^2-2}{(1+x^2)^3}$$

$$f''(0) = \frac{0-2}{1^3} = -2$$

$\boxed{-2 < 0 \Rightarrow x = 0 \text{ IS A LOCAL MAXIMUM}}$

C) $f(x) = x^2 - 2x + 1$

$f'(x) = 2x - 2$ $\qquad f''(x) = 2$

$f'(x_0) = 0$ $\qquad f''(1) = 2$

$2x - 2 = 0$

$2x = 2$ $\qquad \boxed{2 > 0 \Rightarrow x = 1 \text{ IS A LOCAL MINIMUM}}$

$x = 1$

D) $f(x) = 2x\tan(x)$

$$f'(x) = \frac{d}{dx}(2x)\cdot\tan(x) + 2x\cdot\frac{d}{dx}(\tan(x))$$

$$-\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$f'(x) = 2\tan(x) + 2x\sec^2(x)$$
$$f'(x) = 2(\tan(x) + x\sec^2(x))$$
$$f'(x_0) = 0$$
$$2(\tan(x) + x\sec^2(x)) = 0$$
$$\tan(x) + x\sec^2(x) = 0$$
$$x = 0$$

$$f''(x) = \frac{d}{dx}(2(\tan(x) + x\sec^2(x)))$$

$$f''(x) = 4(x\tan(x)+1)\sec^2(x)$$
$$f''(0) = 4(0+1)(1)$$
$$f''(0) = 4$$

$\boxed{4 > 0 \Rightarrow x=0 \text{ IS A LOCAL MINIMUM}}$

* PROBLEM 11.3

A) $f(x) = x^3 + ax^2 + bx + c$

$$f'(x) = 3x^2 + 2ax + b$$
$$f''(x) = 6x + 2a$$
$$6x + 2a = 0$$
$$6x = -2a$$

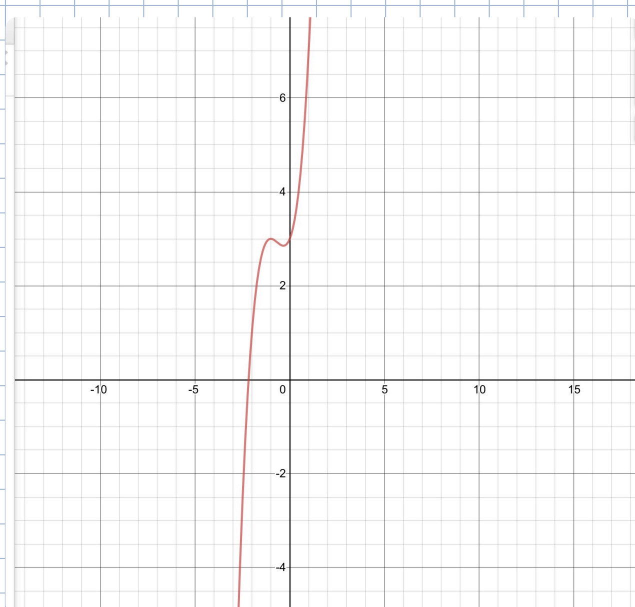
$$\boxed{x = -\frac{1}{3}a}$$

B) $f(x) = x^3 + ax^2 + bx + c$

$a = 2$

$b = 1$

$c = 3$



$$f(x) = x^3 + 2x^2 + x + 3$$
$$f'(x) = 3x^2 + 4x + 1$$
$$f'(x_0) = 0$$
$$3x^2 + 4x + 1 = 0$$
$$x_1 = -1$$
$$x_2 = -\frac{1}{3}$$

$$f''(x) = 6x + 4$$
$$f''(-1) = -6 + 4$$
$$f''(-1) = -2$$

$$f''\left(-\frac{1}{3}\right) = -2 + 4$$
$$f''\left(-\frac{1}{3}\right) = 2$$

$-2 < 0 \Rightarrow x = -1$ IS A LOCAL MAXIMUM

$2 > 0 \Rightarrow x = -\frac{1}{3}$ IS A LOCAL MINIM

$$f''(x) = 6x + 4$$
$$6x + 4 = 0$$
$$6x = -4$$
$$x = -\frac{4}{6}$$

$\boxed{x = -\frac{2}{3} \Rightarrow \text{THIS IS THE INFLECTION POINT AND IT NOT A CRITICAL FUNCTION}}$

* PROBLEM 11.4

$f(x) = x^4 - cx^2$

A) $c = 1$

$f(x) = x^4 - x^2$

$f'(x) = 4x^3 - 2x$

$f'(x_0) = 0$

$4x^3 - 2x = 0$

$x_1 = -\frac{1}{\sqrt{2}} \quad x_2 = \frac{1}{\sqrt{2}} \quad x_3 = 0$

$f''(0) = 12x^2 - 2$

$f''\left(-\frac{1}{\sqrt{2}}\right) = 12 \cdot \left(-\frac{1}{\sqrt{2}}\right)^2 - 2 = 12 \cdot \frac{1}{2} - 2 = 6 - 2 = 4$

$f''\left(\frac{1}{\sqrt{2}}\right) = 12 \cdot \left(\frac{1}{\sqrt{2}}\right)^2 - 2 = 12 \cdot \frac{1}{2} - 2 = 6 - 2 = 4$

$f''(0) = 12 \cdot 0 - 2 = 0 - 2 = -2$

$4 > 0 \Rightarrow x = -\frac{1}{\sqrt{2}}$ IS A LOCAL MINIMUM

$4 > 0 \Rightarrow x = \frac{1}{\sqrt{2}}$ IS A LOCAL MINIMUM

$-2 < 0 \Rightarrow x = 0$ IS A LOCAL MAXIMUM

B) $c = -1$

$f(x) = x^4 - cx^2$

$f(x) = x^4 + x^2$

$f'(x) = 4x^3 + 2x$

$f'(x_0) = 0$

$4x^3 + 2x = 0$

$x = 0$

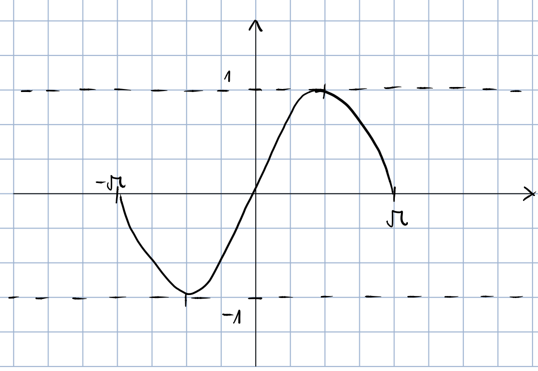
$f''(x) = 12x^2 + 2$

$f''(0) = 2$

$2 > 0 \Rightarrow x = 0$ IS A LOCAL MINIMUM

* PROBLEM 11.5

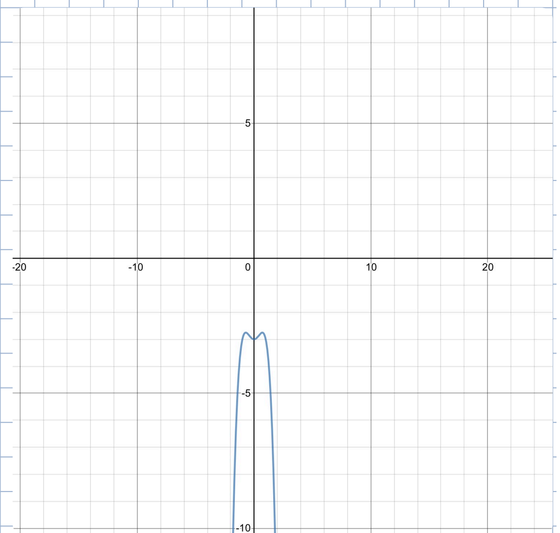
A) ONE LOCAL MAXIMUM AND ONE LOCAL MINIMUM



$\boxed{f(x) = \sin(x)}$ $x \in [-\pi, \pi]$

B) TWO LOCAL MAXIMUM AND ONE LOCAL MINIMUM

$$\boxed{\begin{aligned} f(x) &= -(x^4 - x^2 + 3) \\ f(x) &= -x^4 + x^2 - 3 \end{aligned}}$$



$f'(x) = -4x^3 + 2x$

$f'(x_0) = 0$

$-4x^3 + 2x = 0$

$x_1 = -\frac{1}{\sqrt{2}} \quad x_2 = \frac{1}{\sqrt{2}} \quad x_3 = 0$

$f''(x) = -12x^2 + 2$

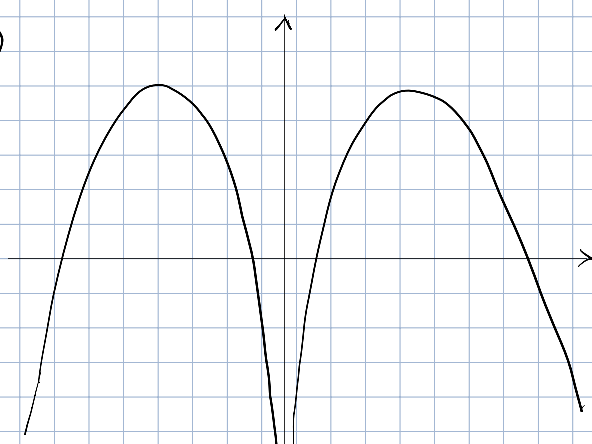
$f''\left(-\frac{1}{\sqrt{2}}\right) = -12 \cdot \frac{1}{2} + 2 = -6 + 2 = -4 \Rightarrow$ MAXIMUM

$f''\left(\frac{1}{\sqrt{2}}\right) = -12 \cdot \frac{1}{2} + 2 = -6 + 2 = -4 \Rightarrow$ MAXIMUM

$f''(0) = -12 \cdot 0 + 2 = 2 \Rightarrow$ MINIMUM

C) IF A FUNCTION IS DIFFERENTIABLE AND HAS 2 LOCAL MAXIMAS IT HAS TO HAVE A LOCAL MINIMUM BETWEEN THEM, OTHERWISE IT WOULD NOT BE DIFFERENTIABLE.

D)



• A FUNCTION THAT IS NOT DEFIRENTIABLE EVERYWHERE, HAS 2 LOCAL MAXIMA AND NO LOCAL MINIMA