MATH 1A – PROBLEM SET 13

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* PROBLEM 13.1

A) $f(x)=\dfrac{9x}{\sin(3x)}$ $\qquad \lim_{x\to 0} f(x) = \lim_{x\to 0}\dfrac{9x}{\sin(3x)} = \lim_{x\to 0}\dfrac{9}{3\cos(3x)} = \dfrac{9}{3} = 3$

$$\boxed{\lim_{x\to 0}\frac{9x}{\sin(3x)} = 3}$$

B) $f(x)=\dfrac{(\exp(6x)-1)}{(\exp(7x)-1)}$ $\qquad \lim_{x\to 0} f(x) = \lim_{x\to 0}\dfrac{(\exp(6x)-1)}{(\exp(7x)-1)} = \lim_{x\to 0}\dfrac{6e^{6x}}{7e^{7x}} = \dfrac{7}{6}$

$$\boxed{\lim_{x\to 0}\frac{(\exp(6x)-1)}{(\exp(7x)-1)} = \frac{6}{7}}$$

C) $f(x)=\dfrac{\sin^2(4x)}{\sin^2(5x)}$ $\qquad \lim_{x\to 0} f(x) = \lim_{x\to 0}\dfrac{\sin^2(4x)}{\sin^2(5x)} = \lim_{x\to 0}\left(\dfrac{\sin(4x)}{\sin(5x)}\right)^2 = \lim_{x\to 0}\left(\dfrac{4\cos(4x)}{5\cos(5x)}\right)^2 = \lim_{x\to 0}\left(\dfrac{4}{5}\right)^2$

$$\boxed{\lim_{x\to 0}\frac{\sin^2(4x)}{\sin^2(5x)} = \frac{16}{25}}$$

D) $f(x)=\dfrac{\sin(x^2)}{\sin^2(x)}$ $\qquad \lim_{x\to 0} f(x) = \lim_{x\to 0}\dfrac{\sin(x^2)}{\sin^2(x)} = \lim_{x\to 0}\dfrac{2x\cos(x^2)}{2\sin(x)\cos(x)} = \lim_{x\to 0}\dfrac{2x\cos(x^2)}{\sin(2x)}$

$$= \lim_{x\to 0}\frac{2(\cos(x^2)-2x^2\sin(x^2))}{2\cos(2x)}$$

$\dfrac{d}{dx}(\sin(x^2)) = \dfrac{d}{du}(\sin(u))\cdot\dfrac{d}{dx}(u)$ $\qquad u=x^2$

$= \cos(u)\cdot 2x$

$= 2x\cos(x^2)$

$$\boxed{\lim_{x\to 0}\frac{\sin(x^2)}{\sin^2(x)} = 1}$$

$\dfrac{d}{dx}(\sin^2(x)) = \dfrac{d}{du}(u^2)\cdot\dfrac{d}{dx}(u)$ $\qquad u=\sin(x)$

$= 2u \cdot \cos(x)$

$= 2\sin(x)\cos(x)$

$\dfrac{d}{dx}(2x\cos(x^2)) = \dfrac{d}{dx}(2x)\cdot\cos(x^2) + 2x\cdot\dfrac{d}{dx}(\cos(x^2))$ $\qquad \dfrac{d}{dx}(\sin(2x)) = 2\cos(2x)$

$= 2(\cos(x^2) - 2x^2\sin(x^2))$

E) $f(x)=\dfrac{\sin(\sin(11x))}{x}$ $\qquad \lim_{x\to 0} f(x) = \lim_{x\to 0}\dfrac{\sin(\sin(11x))}{x} = \lim_{x\to 0}\dfrac{11\cos(11x)\cos(\sin(11x))}{1}$

$\dfrac{d}{dx}(\sin(\sin(11x))) = \dfrac{d}{du}(\sin(u))\cdot\dfrac{d}{dx}(u)$ $\qquad u=\sin(11x)$

$$\boxed{\lim_{x\to 0}\frac{\sin(\sin(11x))}{x} = 11}$$

$= \cos(u)\cdot 11\cos(11x)$

$= \cos(\sin(11x))\cdot 11\cos(11x)$

$= 11\cos(11x)\cos(\sin(11x))$

* PROBLEM 13.2

A) $f(x) = \dfrac{\sin(\sqrt{\pi}x)}{x}$ $\qquad \lim\limits_{x\to 1} f(x) = \boxed{\lim\limits_{x\to 1} \dfrac{\sin(\sqrt{\pi}x)}{x} = 0}$

B) $f(x) = \dfrac{(x-1)^2}{\cos(x-1)-1}$ $\qquad \lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{(x-1)^2}{\cos(x-1)-1} = \lim\limits_{x\to 1} \dfrac{2x-2}{-\sin(x-1)} = \lim\limits_{x\to 1} -\dfrac{2}{\cos(x-1)}$

$\dfrac{d}{dx}(x-1)^2 = \dfrac{d}{x}(x^2-2x+1)$

$= 2x-2$

$$\boxed{\lim_{x\to 1} \frac{(x-1)^2}{\cos(x-1)-1} = -2}$$

$\dfrac{d}{dx}(\cos(x-1)) = \dfrac{d}{du}(\cos(u)) \cdot \dfrac{d}{dx}(u) \quad u=x-1$ $\qquad \dfrac{d}{dx}(2x-2)=2$ $\qquad \dfrac{d}{dx}(-\sin(x-1)) = -\cos(x-1)$

$= -\sin(u) \cdot 1$

$= -\sin(x-1)$

C) $f(x) = \dfrac{(\exp(x)-e)}{(\exp(3x)-e^3)}$ $\qquad \lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{(\exp(x)-e)}{(\exp(3x)-e^3)} = \lim\limits_{x\to 1} \dfrac{e^x}{3e^{3x}} = \dfrac{\cancel{e}}{3e^{\cancel{3}2}}$

$$\boxed{\lim_{x\to 1} \frac{(\exp(x)-e)}{(\exp(3x)-e^3)} = \frac{1}{3e^2}}$$

D) $f(x) = \dfrac{(x-4)}{(4x+\sin(\sqrt{\pi}x))}$ $\qquad \lim\limits_{x\to 1} f(x) = \boxed{\lim\limits_{x\to 1} \dfrac{(x-4)}{4x+\sin(\sqrt{\pi}x)} = \dfrac{-3}{4}}$

E) $f(x) = \dfrac{(x^2+x-1)}{\sqrt{4x^4+1}}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{(x^2+x-1)}{\sqrt{4x^4+1}} = \lim_{x\to\infty} \sqrt{\frac{(x^2+x-1)^2}{4x^4+1}} = \lim_{x\to\infty} \sqrt{\frac{2(2x+1)(x^2+x-1)}{16x^3}}$$

$$\frac{d}{dx}(x^2+x-1)^2 = \frac{d}{du}(u^2)\cdot\frac{d}{dx}(u) \qquad u = x^2+x-1$$

$$= 2u\cdot(2x+1)$$

$$= 2(x^2+x-1)(2x+1) = (2x^2+2x-2)(2x+1)$$

$$= 4x^3+2x^2+4x^2+2x-4x-2$$

$$= 4x^3+6x^2-2x-2$$

$$\frac{d}{dx}(4x^4+1) = 16x^3$$

$$\frac{d}{dx}(4x^3+6x^2-2x-2) = 12x^2+12x-2$$

$$\frac{d}{dx}(16x^3) = 48x^2$$

$$= \lim_{x\to\infty} \sqrt{\frac{4x^3+6x^2-2x-2}{16x^3}}$$

$$= \lim_{x\to 0} \sqrt{\frac{12x^2+12x-2}{48x^2}}$$

$$= \lim_{x\to 0} \sqrt{\frac{24x+12}{96x}}$$

$$= \lim_{x\to 0} \sqrt{\frac{24}{96}}$$

$$= \lim_{x\to 0} \sqrt{\frac{1}{4}}$$

$$\boxed{\lim_{x\to 0} \frac{(x^2+x-1)^2}{\sqrt{4x^4+1}} = \frac{1}{2}}$$

* PROBLEM 13.3

A) $f(x) = \dfrac{\log|5x|}{\log|x|}$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\log|5x|}{\log|x|} = \lim_{x\to 0} \frac{\frac{5}{5x}}{\frac{1}{x}} = \frac{5\cancel{x}}{5\cancel{x}}$$

$$\boxed{\lim_{x\to 0} \frac{\log|5x|}{\log|x|} = 1}$$

B) $f(x) = \dfrac{x}{\log|x|}$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x}{\log|x|} = \lim_{x\to 0} \frac{\frac{1}{1}}{\frac{1}{x}}$$

$$\boxed{\lim_{x\to 0} \frac{x}{\log|x|} = 0}$$

C) $f(x) = 4\,\mathrm{sinc}'(x) = \dfrac{4(\cos(x)x - \sin(x))}{x^2}$ $\qquad \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{4(\cos(x)x - \sin(x))}{x^2} = \lim\limits_{x\to 0} \dfrac{-\cancel{x}\sin(x)}{2\cancel{x}}$

$$\frac{d}{dx}\left(4(\cos(x)x - \sin(x))\right) = \frac{d}{dx}(\cos(x)) \cdot x + \cos(x)\cdot 1 - \sin(x))$$

$$\boxed{\lim_{x\to 0} \frac{4(\cos(x)x - \sin(x))}{x^2} = 0}$$

$$= -\sin(x)\cdot x + \cancel{\cos}(x) - \cancel{\cos}(x)$$

$$= -\sin(x)\cdot x$$

$$\frac{d}{dx}(x^2) = 2x$$

D) $f(x) = \dfrac{\log|1+x|}{\log|\log|1+x||}$ $\qquad \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{\log|1+x|}{\log|\log|1+x||} = \lim\limits_{x\to 0} \dfrac{\frac{1}{1+\cancel{x}}}{\frac{1}{\log|x+1|\cdot(1+\cancel{x})}} = \lim\limits_{x\to 0} \log|x+1|$

$$\frac{d}{dx}(\log|1+x|) = \frac{1}{1+x}$$

$$\boxed{\lim_{x\to 0} \frac{\log|1+x|}{\log|\log|1+x||} = 0}$$

$$\frac{d}{dx}(\log|\log|1+x||) = \frac{d}{du}(\log(u)) \cdot \frac{d}{dx}(u) \qquad u = \log|x+1|$$

$$= \frac{1}{u} \cdot \frac{1}{1+x} = \frac{1}{\log|x+1|} \cdot \frac{1}{1+x}$$

E) $f(x) = \dfrac{(e^x - 1)}{(e^{2x} - 1)}$ $\qquad \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{e^x - 1}{e^{2x} - 1} = \lim\limits_{x\to 0} \dfrac{e^x}{2e^{2x}}$

$$\boxed{\lim_{x\to 0} \frac{e^x - 1}{e^{2x} - 1} = \frac{1}{2}}$$

* PROBLEM 13.4

A) $f(x) = \dfrac{x^{1000} - 1}{x^{40} - 1}$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^{1000} - 1}{x^{40} - 1} = \lim_{x \to 1} \frac{1000x^{999}}{40x^{39}}$$

$$\boxed{\lim_{x \to 1} \frac{x^{1000} - 1}{x^{40} - 1} = 25}$$

B) $f(x) = \dfrac{\tan^2(x-1)}{(\cos(x-1) - 1)}$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{\tan^2(x-1)}{\cos(x-1) - 1} = \lim_{x \to 1} \frac{\dfrac{2\tan(x-1)}{\cos^2(x-1)}}{\dfrac{-\sin(x-1)}{1}} = \lim_{x \to 1} - \frac{2\tan(x-1)}{\cos^2(x-1)\sin(x-1)}$$

$$\frac{d}{dx}(\tan^2(x-1)) = \frac{d}{du}(u^2) \cdot \frac{d}{dx}(u) \quad u = \tan(x-1)$$

$$= 2\tan(x-1) \cdot \frac{1}{\cos^2(x-1)}$$

$$= \frac{2\tan(x-1)}{\cos^2(x-1)}$$

$$= \lim_{x \to 1} \frac{-\dfrac{2}{1} \cdot \left(\dfrac{\cancel{\sin}(x-1)}{\cos(x-1)}\right)}{\dfrac{\cos^2(x-1)\cancel{\sin}(x-1)}{1}}$$

$$= \lim_{x \to 1} \frac{-2}{\cos(x-1)\cos^2(x-1)}$$

$$\boxed{\lim_{x \to 1} \frac{\tan^2(x-1)}{\cos(x-1) - 1} = -2}$$

$$\frac{d}{dx}(\cos(x-1) - 1) = \frac{d}{du}(\cos(u)) \cdot \frac{d}{dx}(u) \quad u = x - 1$$

$$= -\sin(u) \cdot 1$$

$$= -\sin(x-1)$$

* PROBLEM 13.5

A) $f(x) = \frac{x^2-25}{x-5}$ $\lim_{x\to 5} f(x) = \lim_{x\to 5} \frac{x^2-25}{x-5} = \lim_{x\to 5} \frac{\cancel{(x-5)}(x+5)}{\cancel{(x-5)}}$

$$\boxed{\lim_{x\to 5} \frac{x^2-25}{x-5} = 10}$$

B) $f(x) = \frac{1-e^x}{x-x^3}$ $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1-e^x}{x-x^3} = \frac{-e^x}{1-x^2}$

$\frac{d}{dx}(1-e^x) = e^x$

$$\boxed{\lim_{x\to 0} \frac{1-e^x}{x-x^3} = -1}$$

$\frac{d}{dx}(x-x^3) = 1-x^2$

C) $f(x) = \frac{\log(1+9x)}{4x}$ $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\log(1+9x)}{4x} = \lim_{x\to 0} \frac{\frac{9}{1+9x}}{\frac{4}{1}} = \lim_{x\to 0} \frac{9}{4(1+9x)}$

$\frac{d}{dx}\log(1+9x) = \frac{1}{1+9x} \cdot 9 = \frac{9}{1+9x}$

$$\boxed{\lim_{x\to 0} \frac{\log(1+9x)}{4x} = \frac{9}{4}}$$

$\frac{d}{dx}(4x) = 4$

D) $f(x) = \frac{(x^7-1)}{(x^3-1)}$ $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{(x^7-1)}{(x^3-1)} = \lim_{x\to 1} \frac{7x^6}{3x^2}$

$\frac{d}{dx}(x^7-1) = 7x^6$

$$\boxed{\lim_{x\to 1} \frac{x^7-1}{x^3-1} = \frac{7}{3}}$$

$\frac{d}{dx}(x^3-1) = 3x^2$

E) $f(x) = \frac{13x}{\tan(6x)}$ $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{13x}{\tan(6x)} = \lim_{x\to 0} \frac{\frac{13}{1}}{\frac{6}{\cos^2(6x)}} = \lim_{x\to 0} \frac{13\cos^2(6x)}{6}$

$\frac{d}{dx}(13x) = 13$

$$\boxed{\lim_{x\to 0} \frac{13x}{\tan(6x)} = \frac{13}{6}}$$

$\frac{d}{dx}(\tan(6x)) = \frac{d}{du}(\tan(u)) \cdot \frac{d}{dx}(u)$ $u = 6x$

$= \frac{1}{\cos^2(6x)} \cdot 6 = \frac{6}{\cos^2(6x)}$