MATH 1A - PROBLEM SET 21

* PROBLEM 1

- PARABOLOID

$4 - x^2$ $\quad$ $A(x) = x^2\pi$

$x \in (0, 2)$ $\quad$ $A(4-x^2) = (4-x^2)^2\pi$

$A(4-x^2) = (16 - 8x^2 + x^4)\pi$

$$V = \int_a^b A(4-x^2)\,dx$$

$$= \int_0^2 \pi(16 - 8x^2 + x^4)\,dx$$

$$= \int_0^2 \pi(x^4 - 8x^2 + 16)\,dx$$

$$= \int_0^2 \pi x^4 - \pi 8x^2 + 16\pi\,dx$$

$$= \int_0^2 \pi x^4\,dx - \int_0^2 \pi 8x^2\,dx + \int_0^2 16\pi\,dx$$

$$= \pi \int_0^2 x^4\,dx - 8\pi \int_0^2 x^2\,dx + 16\pi \int_0^2 1\,dx$$

$$= \pi \cdot \frac{x^5}{5} - 8\pi \frac{x^3}{3} + 16\pi \cdot x \Big|_0^2$$

$$= \frac{\pi x^5}{5} - \frac{8\pi x^3}{3} + 16\pi x \Big|_0^2 = \boxed{\frac{256\pi}{15}}$$

$$\int_0^2 \pi(16 - 8x^2 + x^4)\,dx = \boxed{\frac{256\pi}{15}}$$

* PROBLEM 2

$\Rightarrow$ CATENOID $(-2, 2)$

$$f(x) = \cosh(x) = \frac{\exp(x) + \exp(-x)}{2}$$

$$\cosh^2(x) = \frac{1 + \cosh(2x)}{2}$$

$$A(x) = \pi f(x)^2$$
$$A(x) = \pi \cosh^2(x)$$

$$A(x) = \frac{\pi(1 + \cosh(2x))}{2}$$

$$V = \int_a^b A(x)\, dx$$
$$= \int_{-2}^{2} \pi \cosh^2(x)\, dx$$
$$= \int_{-2}^{2} \frac{\pi(1 + \cosh(2x))}{2}\, dx$$
$$= \int_{-2}^{2} \frac{\pi}{2} + \frac{\pi \cosh(2x)}{2}\, dx$$
$$= \int_{-2}^{2} \frac{\pi}{2}\, dx + \int_{-2}^{2} \frac{\pi \cosh(2x)}{2}\, dx$$
$$= \frac{\pi}{2} \int_{-2}^{2} 1\, dx + \frac{\pi}{2} \int_{-2}^{2} \cosh(2x)\, dx$$
$$= \frac{\pi}{2} \int_{-2}^{2} 1\, dx + \frac{\pi}{4} \int_{-2}^{2} \cosh(u)\, dx$$
$$= \frac{\pi}{2} \cdot x + \frac{\pi}{2} \cdot \frac{\sinh(2x)}{2} \Big|_{-2}^{2}$$
$$= \frac{\pi x}{2} + \frac{\pi \sinh(2x)}{4} \Big|_{-2}^{2}$$
$$= 2\pi + \frac{\pi \sinh(4)}{2} = \boxed{49.15}$$

$$\int_{-2}^{2} \pi \cosh^2(x)\, dx = \boxed{49.15}$$

* PROBLEM 3

$\Rightarrow$ TOMATO

$$z + x^2 + 4y^2 = 1$$

$$\text{SLICE} = z^2 + x^2 \leq 1 - 4y^2$$

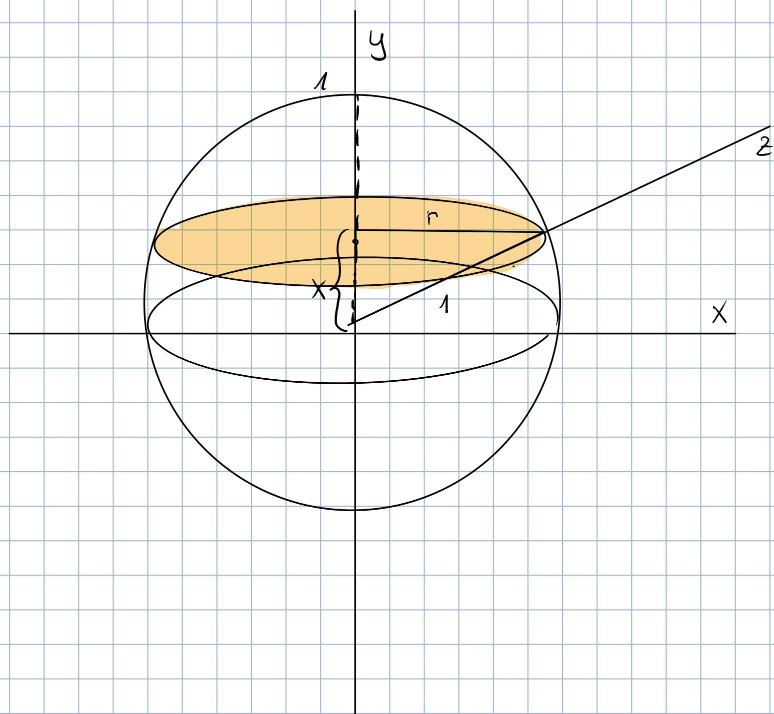
$$\text{RADIUS} = \sqrt{1 - 4y^2}$$

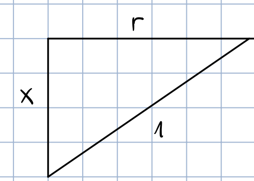
$$r = \sqrt{1 - 4y^2}$$
$$1 - 4y^2 = 0$$
$$-4y^2 = 1$$
$$4y^2 = 1$$
$$y^2 = \frac{1}{4}$$
$$y = \pm \frac{1}{2}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \pi(1 - 4y^2)\, dy = \pi \int_{-\frac{1}{2}}^{\frac{1}{2}} 1 - 4y^2\, dy$$

$$\pi \left(y - \frac{4}{3} y^3 \right) \Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \boxed{\frac{2\pi}{3} \simeq 2.094}$$

* PROBLEM 4





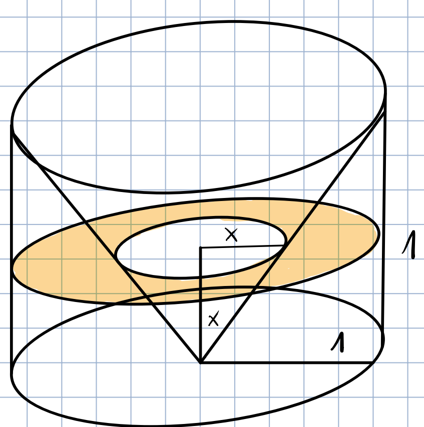
$r = \sqrt{1 - x^2}$

$A(x) = \pi(1 - x^2)$

$\int_{-1}^{1} \pi(1 - x^2)\,dx$

$\left(x - \frac{x^3}{3}\right)\Big|_{-1}^{1} = \frac{4\pi}{3}$ SPHERE

HALF SPHERE $\frac{2\pi}{3}$



IF SLICE IS THE SAME AT EVERY HEIGHT, THE VOLUME WILL ALSO BE THE SAME

$A(x) = \pi - \pi z^2$

CYLINDER − CONE = HALF SPHERE

$\pi - \frac{\pi}{3} = \frac{2\pi}{3}$

* PROBLEM 5

$$\boxed{V(\text{PYRAMID}) = \frac{h(a^2+ab+b^2)}{3}}$$

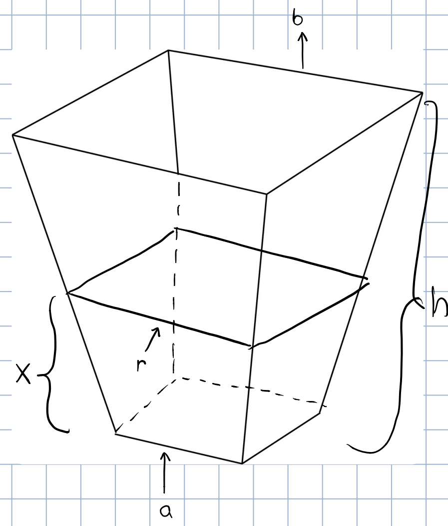
a = BASE LENGTH
b = ROOF LENGTH
h = HEIGHT

$$A(x) = \left(a + (b-a)\frac{x}{h}\right)^2$$

$$r = a + (b-a)\frac{x}{h}$$

$\boxed{\text{if } x=0}$ $r = a + (b-a)\cdot 0$
$r = a$

$\boxed{\text{if } x=h}$ $r = a + (b-a)\cdot 1$
$r = b$



$$V(\text{PYRAMID}) = \int_0^h \left(a + (b-a)\frac{x}{h}\right)^2$$

$$\int_0^h (u)^2 \qquad u = a + (b-a)\frac{x}{h}$$

$$\frac{h}{b-a}\int u^2\,du$$

$$\frac{h\,u^3}{(b-a)3} = \left.\frac{h\left(\frac{(b-a)x}{h}+a\right)^3}{3(b-a)}\right|_0^h$$

$$= \frac{h\left((b-a)+a\right)^3}{3(b-a)} - \frac{h(a)^3}{3(b-a)}$$

$$= \frac{h(b^3)}{3(b-a)} - \frac{h(a^3)}{3(b-a)} = \frac{h(b^3-a^3)}{3(b-a)}$$

$$= \frac{h\cancel{(b-a)}(a^2+ab+b^2)}{3\cancel{(b-a)}}$$

$$= \boxed{\frac{h(a^2+ab+b^2)}{3}}$$

$$V(\text{PYRAMID}) = \int_0^h \left(a + (b-a)\frac{x}{h}\right)^2 = \boxed{\frac{h(a^2+ab+b^2)}{3}}$$