MATH 1A - PROBLEM SET 24

$$\boxed{f(x) = g'(u(x))\,u'(x) \qquad F(x) = G(u(x)) + C}$$

* PROBLEM 1

A) $\displaystyle\int 20x\sin(x^2)\,dx = \int \sin(x^2)\,20x\,dx$

$u = x^2$

$du = 2x\,dx$

$dx = \dfrac{du}{2x}$

$\displaystyle = \int \sin(u)\,\cancel{20x}\,\frac{du}{\cancel{2x}}$

$\displaystyle = \int 10\sin(u)\,du$

$= \boxed{-10\cos(x^2) + C}$

B) $\displaystyle\int e^{x^6+x}(6x^5+1)\,dx = \int e^u(6x^5+1)\frac{du}{6x^5+1}$

$u = x^6 + x$

$du = 6x^5 + 1\ dx$

$dx = \dfrac{du}{6x^5+1}$

$\displaystyle = \int e^u\,du$

$= e^u + C$

$= \boxed{e^{x^6+x} + C}$

C) $\displaystyle\int \cos(\cos^3(x))\sin(x)\cos^2(x)$

$u = \cos^3(x)$

$du = -3\cos^2(x)\cdot\sin(x)\ dx$

$dx = \dfrac{du}{-3\cos^2(x)\sin(x)}$

$\displaystyle\int \cos(u)\,\cancel{\sin(x)}\,\cancel{\cos^2(x)}\,\frac{du}{-3\cancel{\cos^2(x)}\,\cancel{\sin(x)}}$

$\displaystyle -\frac{1}{3}\int \cos(u)\,du$

$= -\dfrac{\sin(u)}{3} + C = \boxed{\dfrac{-\sin(\cos^3(x))}{3} + C}$

D) $\displaystyle\int \frac{e^{\tan(x)}}{\cos^2(x)}$

$u = \tan(x)$

$du = \sec^2(x)\,dx$

$du = \dfrac{1}{\cos^2(x)}\,dx \qquad dx = \dfrac{du}{\dfrac{1}{\dfrac{1}{\cos^2(x)}}} = du\cdot\cos^2(x)$

$\displaystyle\int \frac{e^u}{\cancel{\cos^2(x)}}\,du\cdot\cancel{\cos^2(x)}$

$\displaystyle\int e^u\,du = e^u + C = \boxed{e^{\tan(x)} + C}$

* PROBLEM 2

A) $\displaystyle\int_2^5 \sqrt{x^5+x}\,\left(x^4+\frac{1}{5}\right)\,dx$

$u = x^5 + x$

$du = 5x^4 + 1 \; dx$

$dx = \dfrac{du}{5x^4+1}$

$$\int_2^5 \sqrt{u}\,\left(x^4+\frac{1}{5}\right)\frac{du}{5x^4+1}$$

$$\int_2^5 \sqrt{u}\,\left(\cancel{x^4+\frac{1}{5}}\right)\frac{du}{5\left(\cancel{x^4+\frac{1}{5}}\right)}$$

$$\frac{1}{5}\int_2^5 \sqrt{u}\;du$$

$$= \frac{u^{3/2}\,2}{15}\Bigg|_2^5 = \frac{(x^5+x)^{\frac{3}{2}}\cdot 2}{15} + C\;\Bigg|_2^5$$

$$= \frac{(5^5+5)^{\frac{3}{2}}\cdot 2}{15} - \frac{(2^5+2)^{\frac{3}{2}}\cdot 2}{15}$$

$$= \boxed{\frac{2\left(3310^{\frac{3}{2}} - 34^{\frac{3}{2}}\right)}{15} = 23321.865}$$

B) $\displaystyle\int_0^{\sqrt{\pi}} \sin(x^2)\,x\,dx$

$u = x^2$

$du = 2x\,dx$

$dx = \dfrac{du}{2x}$

$$\int_0^{\sqrt{\pi}} \sin(u)\,\cancel{x}\,\frac{du}{2\cancel{x}}$$

$$\frac{1}{2}\int_0^{\sqrt{\pi}} \sin(u)\,du$$

$$= \frac{-\cos(u)}{2} + C = \frac{-\cos(x^2)}{2}\Bigg|_0^{\sqrt{\pi}} = \frac{-\cos(\pi)}{2} + \frac{\cos(0)}{2} = \boxed{1}$$

C) $\displaystyle\int_{1/e}^{e} \frac{\sqrt{\log(x)}}{x}\,dx$

$u = \log(x)$

$du = \dfrac{1}{x}\,dx$

$dx = \dfrac{\frac{du}{1}}{\frac{1}{x}} = du\cdot x$

$$\int_{1/e}^{e} \frac{\sqrt{u}}{\cancel{x}}\cdot du\cdot \cancel{x}$$

$$\int_{1/e}^{e} \sqrt{u}\;du$$

$$\frac{u^{3/2}\,2}{3} + C = \frac{\log(x)^{3/2}\,2}{3}\Bigg|_{1/e}^{e} =$$ CAN'T BE EVALUATED OVER REAL NUMBERS

D) $\int_0^1 \frac{5x}{\sqrt{1+x^2}}\, dx$

$u = 1 + x^2$

$du = 2x\, dx$

$dx = \frac{du}{2x}$

$\int_0^1 \frac{5x}{\sqrt{u}} \frac{du}{2x}$

$\int_0^1 \frac{1}{\sqrt{u}} \cdot 5\cancel{x} \frac{du}{2\cancel{x}}$

$\frac{5}{2}\int_0^1 \frac{1}{\sqrt{u}}\, du$

$\frac{5}{2}\int_0^1 u^{-\frac{1}{2}}\, du$

$\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = 5\sqrt{u} + C = 5\sqrt{1+x^2}\Big|_0^1 = 5\sqrt{2} - 5\sqrt{1} = 5\sqrt{2} - 5 = \boxed{2.07}$

* PROBLEM 3

$\int_e^{6e} \frac{dx}{\sqrt{\log(x)}\, x} = \int_e^{6e} \frac{1}{\sqrt{\log(x)}\, x}\, dx$

$u = \log(x)$

$du = \frac{1}{x}\, dx$

$\int_e^{6e} \frac{1}{\sqrt{u}\ \cancel{x}}\, du \cdot \cancel{x}$

$dx = \frac{du}{\frac{1}{\frac{1}{x}}}$

$\int_e^{6e} \frac{1}{\sqrt{u}}\, du$

$dx = du \cdot x$

$\int_e^{6e} u^{(-\frac{1}{2})}\, du$

$\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C = 2\sqrt{u} + C = 2\sqrt{\log(x)}\Big|_e^{6e} = \boxed{2\sqrt{\log(6e)} - 2 = 1.3417}$

* PROBLEM 4

A) $$\int \frac{x^5}{\sqrt{x^2+1}}\,dx$$

$$\int 2x \cdot \frac{x^4}{2\sqrt{x^2+1}}\,dx \qquad u^2 = x^4$$

$$u = x^2$$

$$du = 2x\,dx$$

$$\int \frac{u^2}{\sqrt{u+1}} \cdot \cancel{x} \cdot \frac{du}{2\cancel{x}} \qquad dx = \frac{du}{2x}$$

$$\frac{1}{2}\int \frac{u^2}{\sqrt{u+1}}\,du \qquad v = u+1$$

$$v^2 = (u+1)^2$$

$$dv = da$$

$$\int \frac{(v-1)^2}{\sqrt{v}}\,dv$$

$$\int \left(v^{\frac{3}{2}} - 2\sqrt{v} + \frac{1}{\sqrt{v}}\right)dv$$

$$\int v^{\frac{3}{2}}\,dv - 2\int \sqrt{v}\,dv + \int \frac{1}{\sqrt{v}}\,dv$$

$$= \frac{2v^{\frac{5}{2}}}{5} - \frac{4v^{\frac{3}{2}}}{3} + 2\sqrt{v}$$ => UNDO SUBSITUTION $V = u+1$

$$= \frac{(u+1)^{\frac{5}{2}}}{5} - \frac{2(u+1)^{\frac{3}{2}}}{3} + \sqrt{u+1}$$ => UNDO SUBSITUTION $u = x^2$

$$= \frac{(x^2+1)^{\frac{5}{2}}}{5} - \frac{2(x^2+1)^{\frac{3}{2}}}{3} + \sqrt{x^2+1}$$

$$= \boxed{\frac{\sqrt{x^2+1}\,(3x^4 - 4x^2 + 8)}{15} + C}$$

B) $$\int \frac{1}{x(1+\log(x)^2)}$$

$$\int \frac{1}{\cancel{x}(1+u^2)}\,du\,\cancel{x} \qquad u = \log(x)$$

$$du = \frac{1}{x}\,dx$$

$$\int \frac{1}{1+u^2} \qquad dx = du\,x$$

$$= \arctan(u) = \boxed{\arctan(\log(x))}$$

* PROBLEM 5

$$\int \frac{x^5}{\sqrt{x^2+1}}\,dx$$

$$\int 2x \cdot \frac{x^4}{2\sqrt{x^2+1}}\,dx \qquad u^2 = x^4$$

$u = x^2$

$du = 2x\,dx$

$$\int \frac{u^2}{\sqrt{u^2+1}} \cdot \cancel{x} \cdot \frac{du}{2\cancel{x}} \qquad dx = \frac{du}{2x}$$

$$\frac{1}{2}\int \frac{u^2}{\sqrt{u+1}}\,du \qquad v = u+1$$

$v^2 = (u+1)^2$

$dv = da$

$$\int \frac{(v-1)^2}{\sqrt{v}}\,dv$$

$$\int \left(v^{\frac{3}{2}} - 2\sqrt{v} + \frac{1}{\sqrt{v}}\right) dv$$

$$\int v^{\frac{3}{2}}\,dv - 2\int \sqrt{v}\,dv + \int \frac{1}{\sqrt{v}}\,dv$$

$$= \frac{2v^{\frac{5}{2}}}{5} - \frac{4v^{\frac{3}{2}}}{3} + 2\sqrt{v} \qquad \Rightarrow \text{UNDO SUBSITUTION} \quad v = u+1$$

$$= \frac{(u+1)^{\frac{5}{2}}}{5} - \frac{2(u+1)^{\frac{3}{2}}}{3} + \sqrt{u+1} \qquad \Rightarrow \text{UNDO SUBSITUTION} \quad u = x^2$$

$$= \frac{(x^2+1)^{\frac{5}{2}}}{5} - \frac{2(x^2+1)^{\frac{3}{2}}}{3} + \sqrt{x^2+1}$$

$$= \frac{\sqrt{x^2+1}\,(3x^4 - 4x^2 + 8)}{15}\Bigg|_0^1 = \boxed{\frac{17\sqrt{2}-8}{15} = 0.1266}$$